Metric Dimension of Directed Graphs with Cyclic Covering

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The metric dimension problem was first introduced in 1975 by Slater [6], and independently by Harary and Melter [4] in 1976; however the problem for hypercube was studied (and solved asymptotically) much earlier in 1963 by Erdős and Rényi [2]. A set of vertices $S$ resolves a graph $G$ if every vertex is uniquely determined by its vector of distances to the vertices in $S$. The metric dimension of $G$ is the minimum cardinality of a resolving set of $G$.

An natural analogue for oriented graphs was introduced by Chartrand, Raines, and Zhang musch later in 2000 [1]. Consider an oriented graph $D$. For a vertex $v$ and an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices, the $k$-vector $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$ is referred to as the (directed) representation of $v$ with respect to $W$, where $d(x, y)$ denotes the directed distance from $x$ to $y$. Since a directed $x - y$ path needs not to exist, the vector $r(v|W)$ needs not to exist as well. If $r(v|W)$ exists for every vertex $v$, then the set $W$ is called a resolving set for $D$ if every two distinct vertices have distinct representations. A resolving set of minimum cardinality is called a basis for $D$ and this cardinality is the (directed) dimension $\dim(D)$ of $D$. An oriented graph of dimension $k$ is also called $k$-dimensional.

Since not every oriented graph has a dimension, one fundamental question is the necessary and sufficient conditions for $\dim(D)$ to be defined. The answers are still unknown, although certainly, if $D$ is strong, then $\dim(D)$ is defined. Also, if $D$ is connected and contains a vertex such that $D - v$ is strong, then $\dim(D)$ is defined.

Unlike the undirected version, there are not many results known for directed metric dimension. Characterization of $k$-dimensional oriented graphs is only known for $k = 1$ [1]. Researchers have also studied the directed metric dimension of tournaments [5] and Cayley digraphs [3].

Let $G$ be a graph with cyclic covering. An orientation on $G$ is called $C_n - \text{simple}$ if all directed $C_n$ in the oriented graph are strong. Here we study metric dimension of two simply oriented graphs: the wheels and the fans.

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**Theorem 1.** Let $\vec{W}_n$ be an oriented wheel with $C_3$ – simple orientation, then

(i) $\dim(\vec{W}_3) = 1$;
(ii) $\dim(\vec{W}_n) = 2$ for $n = 4, 5, 6$;
(iii) $\dim(\vec{W}_n) \leq \lfloor \frac{n-1}{2} \rfloor$, for $n \geq 7$.

**Theorem 2.** Let $\vec{F}_{1,n}$ be an oriented fan with $C_3$ – simple orientation, then

(i) $\dim(\vec{F}_{1,n}) = 1$ for $n = 2, 3, 4$;
(ii) $\dim(\vec{F}_{1,n}) \leq \lfloor \frac{n-1}{2} \rfloor$ for $n \geq 5$.

Additionally, we also construct orientations for $k$-dimensional wheels and fans where $k$ is sufficiently small integers.

**Theorem 3.** For sufficiently small $k$, there exist orientations such that wheels and fans are $k$-dimensional.

Lastly, we consider the directed metric dimension of amalgamations of directed cycles. Suppose that $C = \{C_{n_1}, C_{n_2}, \ldots, C_{n_t}\}$ is a collection of $t$ directed cycles. For $n \leq \min\{n_i|1 \leq i \leq t\}$, define the $P_n$-amalgamation of $C$, $P_n - \text{amal}\{C\}$, as the graph formed by taking all the $C_{n_i}$’s and identifying a path $P_n$ in each of the directed cycle.

**Theorem 4.** For $n \leq \min\{n_i|1 \leq i \leq t\}$, $\dim(P_n - \text{amal}\{C\}) = t - 1$.

**References**