An Extremal Problem for Vertex Decomposition of Complete Multipartite Graphs

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A graph is finite and undirected with no multiple edge or loop. Let $\mathcal{H}$ be a family of graphs. For a graph $G$, we call a vertex decomposition $V(G) = V_1 \cup \cdots \cup V_\ell$ an $\mathcal{H}$-decomposition, if $G[V_i] \in \mathcal{H}$ for $1 \leq i \leq \ell$, where $G[V_i]$ is a subgraph of $G$ induced by $V_i$. In the following, we consider the case where $G$ is a complete multipartite graph and $\mathcal{H}$ consists of graphs with a common number of vertices.

Our aim is to find sufficient conditions for the existence of an $\mathcal{H}$-decomposition having some nice properties.

The next result was proved in [1].

**Theorem 1.** Let $k$ and $\ell$ be positive integers. Every complete multipartite graph of order $k_0$ admits a $\{H_1, H_2\}$-decomposition with some complete multipartite graphs $H_1, H_2$ of order $k$.

In the following, $K_{n_1, n_2, \ldots, n_s}$ is denoted by $(n_1, n_2, \ldots, n_s)$. Furthermore, if $t$ partite sets have a common order $a$, we write as $(\ldots, a^t, \ldots)$ instead of $(\ldots, a, a, \ldots, a, \ldots)$.

Let $A_k = \{(a, 1^{k-a}) : 1 \leq a \leq k\}$.

In this paper, we focus on the case $\mathcal{H} \subset A_k$.

**Theorem 2.** Let $k \geq 4$. If $\ell \geq k - 2$, then every complete multipartite graph of order $k\ell$ admits an $A_k$-decomposition.

As a matter of fact, we have a slightly stronger conclusion.

**Proposition 3.** Let $k \geq 4$. If $\ell \geq k - 2$, then every complete multipartite graph of order $k\ell$ admits a $\{(a, 1^{k-a}), (a + 1, 1^{k-(a+1)}), (a + 2, 1^{k-(a+2)})\}$-decomposition with some $a$.

**Outline of the Proof.** Let $\ell = k - 2$ and let $G$ be a complete multipartite graph of order $k\ell$ with partite sets $P_1, \ldots, P_s$.

- Define $a$ as the maximum integer $x$ such that $G$ contains $\ell$ vertex disjoint copies of $(x)$.
- Choose a vertex decomposition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_\ell$ of $G$ such that (1) $G[V_i] \cong (a)$ or $(a + 1)$ for $1 \leq i \leq \ell$, (2) $|V_0|$ is the minimum with respect to

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\(r := \max\{|V_0 \cap P_j| : 1 \leq j \leq s\}\) is the minimum with respect to (1) and (2).

- By the choice of the decomposition, we have \(r \leq \ell\).
- \(V_0\) can be decomposed into \(\ell\) copies of \((1^{k-a})\) and \((1^{k-(a+1)})\).
- Joining copies of \((1^{k-a})\) in \(V_0\) with those of \((a)\) in \(V(G) \setminus V_0\), and copies of \((1^{k-(a+1)})\) in \(V_0\) with those of \((a+1)\) in \(V(G) \setminus V_0\) one by one, we have a required decomposition.

An extremal example for Theorem 2 (and Proposition 3) is \(((k-1)(k-3) - 1, k-2)\). A related result of Proposition 3 is as follows.

**Proposition 4.** Let \(k \geq 4\). If \(\ell \geq 2k-6\), then every complete multipartite graph of order \(k\ell\) admits a \(\{(a, 1^{k-a}), (a+1, 1^{k-(a+1)})\}\)-decomposition with some \(a\).

An extremal example for Proposition 4 is \(((k-1)(k-3)-1, (k-1)(k-3)-1, k-4)\).

In order to make an \(\mathcal{H}\)-decomposition of a given graph \(G\) of order \(k\ell\) under the assumption that \(\mathcal{H}\) consists of graphs of order \(k\), we need \((k) \in \mathcal{H}\) and \((k-1, 1) \in \mathcal{H}\) for \(G = (k\ell - 1, 1)\), and also need \((1^k) \in \mathcal{H}\) for \(G = (1^{k\ell})\). Conversely, these three graphs \((k), (k-1, 1), (1^k)\) suffice for \(\ell\) sufficiently large.

**Proposition 5.** Let \(k \geq 4\). If \(\ell \geq (k-2)^2\), then every complete multipartite graph of order \(k\ell\) admits a \(\{(k), (k-1, 1), (1^k)\}\)-decomposition.

An extremal example for Proposition 5 is \(((k-1)((k-2)^2 - 1) - 1, (k-2)^{k-2}), \text{which has } k-1 \text{ partite sets.}\)

**Proposition 6.** Let \(k \geq 4\). If \(\ell \geq \frac{1}{2}(3k^2 - 9k + 4)\), then every complete multipartite graph of order \(k\ell\) admits a \(\{(k), (k-1, 1)\}\)-decomposition or a \(\{(k), (1^k)\}\)-decomposition.

An extremal example for Proposition 6 is \((k(k-1)(k-3) - 1, (k-1)^2 - 1, (k-2)(k-1) - 1, (k-3)(k-1) - 1, \ldots, 2(k-1) - 1, 1^{(k^2-k-4)/2})\).

**References**