Closure for spanning trees with k-ended stems

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A closure operation is a useful operation in the study of the existence of cycles, paths and other subgraphs in graphs. It was first introduced by Bondy and Chvátal[1].

THEOREM 1. (Bondy and Chvátal[1]) *Let G be a graph and let u and v be two nonadjacent vertices of G.*

(1) Suppose $\deg_G(u) + \deg_G(v) \geq |G|$, *then G has a hamiltonian cycle if and only if $G + uv$ has a hamiltonian cycle.*

(2) Suppose $\deg_G(u) + \deg_G(v) \geq |G| - 1$, *then G has a hamiltonian path if and only if $G + uv$ has a hamiltonian path.*

After [1], many researchers have defined other closure concepts for various graph properties. The interested reader is referred to the survey [2] on closure concepts.

Let T be a tree. An vertex of T, which has degree one, is often called a leaf of T, and the set of leaves of T is denoted by $Leaf(T)$. The subtree $T - Leaf(T)$ of T is called the stem of T and denotes by $Stem(T)$. A spanning tree with specified stem was first considered in [3].

A tree, whose stem has at most k leaves, is called a tree with k-ended stem. For an integer $k \geq 2$ and a graph G, $\sigma_k(G)$ denotes the minimum degree sum of k independent vertices of G. The following theorem gives a sufficient condition using $\sigma_k(G)$ for a graph to have a spanning tree with k-ended stem.

THEOREM 2. (M. Tsugaki and Y. Zhang [4]) *Let G be a connected graph and $k \geq 2$ be an integer. If G satisfies*

$$\sigma_3(G) \geq |G| - 2k + 1,$$

then G has a spanning tree with k-ended stem.

In [4], Kano and the author of this paper give a weaker result for a graph to have a spanning tree with k-ended stem as following:

THEOREM 3. (M. Kano and Z. Yan [5]) *Let G be a connected graph and $k \geq 2$ be an integer. If G satisfies the following one of conditions then G has a spanning tree with k-ended stem.*

(1) $\sigma_{k+1}(G) \geq |G| - k - 1$.

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(2) $\sigma_{k+1}(G) \geq |G| - 2k - 1$, *if* G *is claw-free.*

In this paper, we will show the following theorem.

Theorem 4. *Let* G *be a connected graph and* $k \geq 2$ *be an integer. Let* u *and* v *be a pair of nonadjacent vertices of* G *such that*

$$|N_G(u) \cup N_G(v)| \geq |G| - k - 1.$$

Then G *has a spanning tree with* k*-ended stem if and only if* $G + uv$ *has a spanning tree with* k*-ended stem.*

Before proving Theorem 4, we first show that the condition in Theorem 4 is sharp. Let $k \geq 2$ and $m \geq 1$ be integers, and let K_m be a complete graph. $\{u, u_1, \ldots, u_k, v, v_1, \ldots, v_k\}$ be vertices not containing K_m. Join u, v to all vertices of K_m by edges. Join $u_i (1 \leq i \leq k)$ to u and v_i by edges. Let G denote the resulting graph. Then $G + uv$ has a spanning tree with k-ended stem and $|N_G(u) \cup N_G(v)| = |G| - k - 2$, but G has no spanning tree with k-ended stem.

References

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