Paths and cycles in hypergraphs ♠

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Let $H$ be a $k$–uniform hypergraph on the vertex set $V(H) = \{v_1, v_2, \ldots, v_n\}$ where $n > k$. $v_{n+x}$ with $x \geq 0$ denotes the same vertex as $v_x$ for simplicity of notation. The set of the edges, $k$–element subsets of $V$, is denoted by $E(H) = \{E_1, E_2, \ldots, E_m\}$.

In [4] we gave the following definition:

DEFINITION 1. A cyclic ordering $(v_1, v_2, \ldots, v_n)$ of the vertex set is called a hamiltonian cycle iff for every $1 \leq i \leq n$ there exists an edge $E_j$ of $H$ such that $\{v_i, v_{i+1}, \ldots, v_{i+k-1}\} = E_j$.

Since an ordinary graph is a 2-uniform hypergraph, this definition gives the definition of the hamiltonian cycle in ordinary graphs for $k = 2$. (As a matter of fact, in the original paper the term chain was used instead of cycle, but it seems that everyone prefers cycle.)

The first natural question was to find a Dirac type theorem for hamiltonian cycles. For this, we need to extend the definition of degree for hypergraphs.

DEFINITION 2. The degree of a fixed $l$–tuple of distinct vertices, $\{v_1, v_2, \ldots, v_l\}$, in a $k$–uniform hypergraph is the number of edges of the hypergraph containing all $\{v_1, v_2, \ldots, v_l\}$. It is denoted by $d^{(k)}(v_1, v_2, \ldots, v_l)$. Furthermore $\delta^{(k)}_l(H)$ denotes minimum of $d^{(k)}(v_1, v_2, \ldots, v_l)$ over all $l$–tuples in $H$. (If $k$ is clear from the context, only $\delta_l(H)$ will be used.)

We conjectured that $\delta^{(k)}_{k-1}(H) \geq \left\lfloor \frac{n-k+3}{2} \right\rfloor$ implies the existence of the hamiltonian cycle, and showed that this bound cannot be lowered. From the other side, a Dirac type theorem was proved for any $k$, however, the degree bound was far from being best possible.

THEOREM 1. ([4]) If $H = (V, E)$ is a $k$–uniform hypergraph on $n$ vertices with

$\delta_{k-1}(H) > \left(1 - \frac{1}{2k}\right)n + 4 - k - \frac{2}{k},$

then $H$ contains a hamiltonian cycle.

For $k = 3$ the above result requires roughly $\frac{5}{6}n$ degree bound for each pair of

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vertices, but it is conjectured that only $\frac{1}{2}n$ is needed.

Now, more than a decade later, the problem is nearly settled. In [5] Ruciński, Rödl and Szemerédi proved that the conjecture is asymptotically true for $k = 3$, then in [6] the exact result was given in this case, together with the analogous hamiltonian path result. Finally, in [7] the exact result was given in the general case, however, only for very large hypergraphs.

Meanwhile more than 60 publications appeared in connection with the above notion. There are mostly on different version of Hamiltonian cycles, but also on various extremal questions, including some works of the present author [1, 2, 3]. In my talk I will survey these results.

As far as I know, professor Egawa does not have any publications on hypergraphs yet. However, many of his results deal with questions that may lead to interesting questions in the above settings on hypergraphs. I will pose some open question with this flavor.

References


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