An Extremal Problem for Vertex Decomposition of Complete Multipartite Graphs

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Definition

For a graph $G$ and a family of graphs $\mathcal{H}$, we call a vertex decomposition $V(G) = V_1 \cup \cdots \cup V_\ell$ an $\mathcal{H}$-decomposition, if $G[V_i] \in \mathcal{H}$ for $1 \leq i \leq \ell$, where $G[V_i]$ is a subgraph of $G$ induced by $V_i$. 

In the following, we consider the case where $G$ is a complete multipartite graph and $\mathcal{H}$ consists of graphs with a common number of vertices. Our aim is to find sufficient conditions for the existence of an $\mathcal{H}$-decomposition having some nice properties.
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In the following, we consider the case where $G$ is a complete multipartite graph and $\mathcal{H}$ consists of graphs with a common number of vertices.

Our aim is to find sufficient conditions for the existence of an $\mathcal{H}$-decomposition having some nice properties.
Vertex decomposition of a complete multipartite graph into two kinds of subgraphs of the same order.

**Theorem 1 (Nakamigawa, 2010)**

Let $k$ and $\ell$ be positive integers. Then for any complete multipartite graph $G$ of order $k\ell$, there exists a pair of complete multipartite graphs $H_1, H_2$ of order $k$ such that $G$ admits a $\{H_1, H_2\}$-decomposition.
a complete multipartite graph ⇔ a set of piles of coins

Example of Theorem 1 \((k = 5, \ell = 3)\)

\[ G = (9, 3, 2, 1) \]

\[ H_1 \]

\[ H_1 \]

\[ H_2 \]
Notation and Definition

- $K_{n_1,n_2,...,n_s}$ is denoted by $(n_1, n_2, \ldots, n_s)$.
- Furthermore, if $t$ partite sets have a common order $a$, we write as $(\ldots, a^t, \ldots)$ instead of $(\ldots, a, a, \ldots, a, \ldots)$. 

Let $A_k = \{ (a, 1_k - a) : 1 \leq a \leq k \}$. 

Namely, $A_k$ is a family of graphs of order $k$ which consists of a complete graph, an empty graph and joins of a complete graph and an empty graph.
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  Namely, $A_k$ is a family of graphs of order $k$ which consists of a complete graph, an empty graph and joins of a complete graph and an empty graph.
Main Results

Theorem 2

Let $k \geq 4$. If $\ell \geq k - 2$, then every complete multipartite graph of order $k\ell$ admits an $A_k$-decomposition.
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We will prove a slightly stronger statement.

Proposition 3

Let $k \geq 4$. If $\ell \geq k - 2$, then every complete multipartite graph of order $k\ell$ admits a \{(a, 1^{k-a}), (a + 1, 1^{k-(a+1)}), (a + 2, 1^{k-(a+2)})\}-decomposition with some positive integer $a$. 
Theorem 2

Let $k \geq 4$. If $\ell \geq k - 2$, then every complete multipartite graph of order $k\ell$ admits an $A_k$-decomposition.

An extremal example for Theorem 2 (and Proposition 3) is $G = ((k - 1)(k - 3) - 1, k - 2)$ for $\ell = k - 3$. 
Tightness of Theorem 2

**Theorem 2**

Let \( k \geq 4 \). If \( \ell \geq k - 2 \), then every complete multipartite graph of order \( k\ell \) admits an \( \mathcal{A}_k \)-decomposition.

An extremal example for Theorem 2 (and Proposition 3) is 

\[
G = ((k - 1)(k - 3) - 1, k - 2) \text{ for } \ell = k - 3.
\]

- Since \( G \) has only two partite sets, \( G \) contains no copy of \((a, 1^{k-a})\) for \( a \leq k - 2 \).
- Since \( G \) has at most \( k - 4 \) vertex disjoint copies of \((k - 1)\), \( G \) has no \{\((k), (k - 1, 1)\)\}-decomposition.
Lemma (Clique Decomposition)

Let $\ell$ be a positive integer. Let $G$ be a complete multipartite graph of order $n$ such that the order of every partite set of $G$ is at most $\ell$. Then there exists a vertex decomposition $V(G) = V_1 \cup \cdots \cup V_\ell$ such that $G[V_i] \cong (1^k)$ or $(1^{k+1})$, where $k = \lfloor n/\ell \rfloor$, for $1 \leq i \leq \ell$. 

Example of Lemma (\(\ell = 8\))

\[
G = (7, 7, 5, 3) \\
V_1 \ V_2 \ V_3 \ V_4 \ V_5 \ V_6 \ V_7 \ V_8
\]
Proof of Proposition 3

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Proof. Let $G$ be a complete multipartite graph of order $k\ell$ with partite sets $P_1, \ldots, P_s$. Let us define $a$ as the maximum integer $x$ such that $G$ contains $\ell$ vertex disjoint copies of $(x)$. If $a \geq k - 1$, we have a $\{(k), (k-1, 1)\}$-decomposition of $G$. Hence, we may assume $a \leq k - 2$. 

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Proof.

- Let $G$ be a complete multipartite graph of order $k\ell$ with partite sets $P_1, \ldots, P_s$.
- Let us define $a$ as the maximum integer $x$ such that $G$ contains $\ell$ vertex disjoint copies of $(x)$.
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**Proof.**

- Choose a vertex decomposition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_\ell$ of $G$ such that
  - (1) $G[V_i] \cong (a)$ or $(a + 1)$ for $1 \leq i \leq \ell$, and
  - (2) $|V_0|$ is the minimum with respect to (1).
- Define $r = \max\{|V_0 \cap P_j| : 1 \leq j \leq s\}$.
- **Claim.** $r \leq \ell$. 
Proof of Proposition 3

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- **Claim.** $r \leq \ell$. 

![Diagram of complete multipartite graph decomposition]
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**Proof.**

- By Clique Decomposition Lemma, \( V_0 \) can be decomposed into \( \ell \) copies of \( (1^{k-a}) \) and \( (1^{k-(a+1)}) \).
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Let \( k \geq 4 \). If \( \ell \geq k - 2 \), then every complete multipartite graph of order \( k\ell \) admits a \( \{(a, 1^{k-a}), (a + 1, 1^{k-(a+1)}), (a + 2, 1^{k-(a+2)})\}\)-decomposition with some positive integer \( a \).

Proof.

- By Clique Decomposition Lemma, \( V_0 \) can be decomposed into \( \ell \) copies of \( (1^{k-a}) \) and \( (1^{k-(a+1)}) \).
- Combining a copy of \( (1^{k-a}) \) in \( V_0 \) with a copy of \( (a) \), we have a copy of \( (a, 1^{k-a}) \).
- Combining a copy of \( (1^{k-(a+1)}) \) in \( V_0 \) with a copy of \( (a + 1) \), we have a copy of \( (a + 1, 1^{k-(a+1)}) \) or \( (a + 2, 1^{k-(a+2)}) \). \(\square\)
Proposition 4

Let $k \geq 4$. If $\ell \geq 2k - 6$, then every complete multipartite graph of order $k\ell$ admits a $\{(a, 1^{k-a}), (a + 1, 1^{k-(a+1)})\}$-decomposition with some positive integer $a$. 

The bound of Proposition 4 is tight. An extremal example is $G = (\binom{k-1}{k-3} - 1, \binom{k-1}{k-3} - 1, k-4)$ for $\ell = 2k - 7$.
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- Suppose to the contradiction that $G$ has a $\{(a, 1^{k-a}), (a + 1, 1^{k-(a+1)})\}$-decomposition.
- Since $G$ has only at most three partite sets, we have $a \geq k - 2$.
- Since $G$ contains at most $2k - 8 < \ell$ vertex disjoint copies of $(k - 1)$, we have $a = k - 2$.
- We may assume the 1st partite set is decomposed into $k - 3$ copies of $(k - 2)$ and the 2nd partite set is decomposed into $k - 4$ copies of $(k - 2)$. But the remaining vertices of the 2nd partite set is $2k - 6 > \ell$, there exists no $\{(k - 1, 1), (k - 2, 1^2)\}$-decomposition.
We can decrease the cardinality of $\mathcal{H}$ to three.

**Proposition 5**

Let $k \geq 4$. If $\ell \geq (k - 2)^2$, then every complete multipartite graph of order $k\ell$ admits a $\{(k), (k - 1, 1), (1^k)\}$-decomposition.
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The bound of Proposition 5 is tight. An extremal example is

$$G = ( (k - 1) ((k - 2)^2 - 1) - 1, (k - 2)^{k-2} ),$$

which has $k - 1$ partite sets, for $\ell = (k - 2)^2 - 1$. 
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The bound of Proposition 5 is tight. An extremal example is $G = ((k - 1)((k - 2)^2 - 1) - 1, (k - 2)^{k-2})$, which has $k - 1$ partite sets, for $\ell = (k - 2)^2 - 1$.

- $G$ has no copy of $(1^k)$.
- Since $G$ has at most $\ell - 1$ vertex disjoint copies of $(k - 1)$, there is no $\{(k), (k - 1, 1)\}$-decomposition.
Related Results

The next result is a refinement of Proposition 5.

**Proposition 6**

Let $k \geq 4$. If $\ell \geq \frac{1}{2}(3k^2 - 9k + 4)$, then every complete multipartite graph of order $k\ell$ admits a $\{(k), (k - 1, 1)\}$-decomposition or a $\{(k), (1^k)\}$-decomposition.
Related Results

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**Proposition 6**

Let \( k \geq 4 \). If \( \ell \geq \frac{1}{2}(3k^2 - 9k + 4) \), then every complete multipartite graph of order \( k\ell \) admits a \((k), (k - 1, 1)\)-decomposition or a \((k), (1^k)\)-decomposition.

The bound of Proposition 6 is tight. An extremal example is \( G = (k(k - 1)(k - 3) - 1, (k - 1)^2 - 1, (k - 2)(k - 1) - 1, (k - 3)(k - 1) - 1, \ldots, 2(k - 1) - 1, 1^{(k^2 - k - 4)/2}) \) for \( \ell = (1/2)(3k^2 - 9k + 2) \).
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The bound of Proposition 6 is tight. An extremal example is 
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\]

- Since \( G \) has at most \( \ell - 1 \) vertex disjoint copies of \( (k - 1) \), there is no \( \{(k), (k - 1, 1)\}\)-decomposition.
- The maximum number of vertex disjoint copies of \( (k) \) in \( G \) is \( \ell - (k - 2) \).
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\]

- For \( k - 2 \leq x \leq \ell \), let us remove \( \ell - x \) vertex disjoint copies of \( (k) \) from \( G \), and let us denote the possible maximum number of the remaining vertices over all partite sets by \( r(x) \).
- What we want to show is \( x < r(x) \) for any \( x \) with \( k - 2 \leq x \leq \ell \), and this can be proved by induction on \( x \).
Discussion

How about **graphs** instead of complete multipartite graphs?

**Remark (Hell-Manoussakis-Tuza, 1994)**

*For any integers $k$ and $t$, and sufficiently large multiple $n$ of $k$, every $t$-edge colored complete graph of order $n$ has a vertex decomposition into monochromatic stars on $k$ vertices such that the endpoints of each star induce a monochromatic complete graph of order $k - 1$.***
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- We focus on the case $t = 2$.
- Define $f(k)$ as the minimum integer $\ell$ such that every graph of order $k\ell$ admits a $\{K_k, \overline{K_k}, K_{1,k-1}, \overline{K_{1,k-1}}\}$-decomposition.
- By the above remark, $f(k) < \infty$. 
Discussion

$f(k)$ is defined as the minimum integer $\ell$ such that every graph of order $k\ell$ admits a \{\(K_k, \overline{K_k}, K_{1,k-1}, \overline{K_{1,k-1}}\}\}-decomposition.

Observation

\[c_1 2^{k/2} \leq f(k) \leq c_2 \frac{2^{4k}}{k}.\]
Discussion

$f(k)$ is defined as the minimum integer $\ell$ such that every graph of order $k\ell$ admits a \{\(K_k, \overline{K_k}, K_{1,k-1}, \overline{K_{1,k-1}}\}\}-decomposition.

**Observation**

$$c_1 2^{k/2} \leq f(k) \leq c_2 \frac{2^{4k}}{k}.$$  

**Lower Bound.**

- \((k - 1)\ell + 1 < R(k - 1, k - 1) \Rightarrow \ell < f(k).\)
**Discussion**

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**Observation**

\[c_1 2^{k/2} \leq f(k) \leq c_2 \frac{2^{4k}}{k}.

**Upper Bound.**

- Let $G$ be a graph of order $n = k\ell$ sufficiently large.
- Let $d = R(k, k) - k$.
- Take $d$ vertex disjoint copies of $K_{2k-1}$ or $\overline{K_{2k-1}}$, which may be mixed, from $G$.
- From the remaining vertices, remove copies of $K_k$ or $\overline{K_k}$, until the number of the remaining vertices is $d$.
- By combining a copy of $(K_{2k-1} \text{ or } \overline{K_{2k-1}})$ and a remaining singleton, we can build a copy of $(K_k \text{ or } \overline{K_k})$ and a copy of $(K_{1,k-1} \text{ or } \overline{K_{1,k-1}})$. 
Open Problem

$f(k)$ is defined as the minimum integer $\ell$ such that every graph of order $k\ell$ admits a $\{K_k, \overline{K_k}, K_{1,k-1}, \overline{K_{1,k-1}}\}$-decomposition.

Observation

$c_12^{k/2} \leq f(k) \leq c_2\frac{2^{4k}}{k}$.

Problem

*Find a better upper (or lower) bound of $f(k)$.***