Analytic solutions to Benjamin-Ono Equations

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1. Introduction

We study smoothing effect for the following nonlinear dispersive equation of the Benjamin-Ono type:

$$\begin{cases} \partial_t u + H_x \partial_x^2 u + \partial_x u^2 = 0, & t \in (-T, T), \quad x \in \mathbb{R}, \\ v(0, x) = \phi(x), \end{cases} \tag{1.1}$$

where $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a unknown function and H_x denotes the Hilbert transform defined by $H_x v = \mathcal{F} \frac{\xi}{i|\xi|} \hat{v}$. This equation arises in the water wave theory and u describes long internal gravity wave in deep stratified fluid (see [1], [14]). Our problem here is to investigate a sufficient condition of the initial data ϕ on which the solution has regularizing property up to analyticity.

Our method is based on an operator method which is common to the cases of the KdV equation or nonlinear Schrödinger equations: We introduce the generator of space-time dilation $P = 2t\partial_t + x\partial_x$ that plays a compensating role where the main linear operator $L = \partial_t + H_x \partial_x^2$ can not gain the regularity. As a consequence, we observe analytic smoothing effect for the solution to (1.1) with an initial data having a singularity at one point. Let $H^s = H^s(\mathbb{R})$ be the Sobolev space of order s defined by

$$\|f\|_{H^s} \equiv \|\langle \xi \rangle^s \hat{f}\|_2,$$

where $\hat{f} = \mathcal{F} f$ denotes the Fourier transform of f and $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$.

Theorem 1.1. *Let $s > 3/2$. Suppose that for some $A_0 > 0$, the initial data $\phi \in H^s(\mathbb{R})$ and satisfies*

$$\sum_{k=0}^{\infty} \frac{A_0^k}{k!} \|(x\partial_x)^k \phi\|_{H^s} < \infty,$$

then there exists a unique solution $u \in C(\mathbb{R}, H^s)$ to the nonlinear dispersive equation (1.1) and for any $(t,x) \in \big(\mathbb{R} \setminus \{0\}\big) \times \mathbb{R}$, we have for some $A > 0$

$$|\partial_t^j \partial_x^l u(t,x)| \le C \langle t^{-1} \rangle^{j+l} \langle x \rangle^{2l+3j} A^{j+l} (j+l)!$$

for any $j, l \in \mathbb{N}$. Namely $u(t,\cdot)$ is a real analytic function in both space and time variables for $t \neq 0$.

Remark 1. The assumption on the initial data implies that the data have to be analytic except $x = 0$. On this point the data is assumed to have only H^s regularity. Hence the above theorem states that this singularity disappears after time passed. The weakness of this singularity on the data is depending on the space where we may establish the well-posedness of the equation.

Here we summarize some notation that we would use in what follows. $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$. H^s is the Sobolev space of order s. Let $L = \partial_t + H_x\partial_x^2$ be the linear part of the Benjamin-Ono equation and $P = 2t\partial_t + x\partial_x$ be the dilation operator associated with L. For operators A and B, $[A, B]$ stands for the commutator $AB - BA$. The free propagator group for the linear Benjamin-Ono type evolution is denoted by $e^{-tH_x\partial_x^2}$ which is a unitary operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.

2. Method

In this section we give an overview of the proof and present some difference from the proof of the former cases in [7] and [?]. The results are based on the following observation.

Noting the commutation relation between the generator of the dilation $P = 2t\partial_t + x\partial_x$ and the linear dispersive operator $L \equiv \partial_t + H_x\partial_x^2$:

$$[L, P] = 2L,$$

we have

$$\begin{aligned} LP^k &= (P+2)^k L, \\ (P+2)^k\partial_x &= \partial_x(P+1)^k, \qquad k = 1, 2, \cdots . \end{aligned} \tag{2.1}$$

Applying $P = 2t\partial_x + x\partial_x$ to the equation (1.1) iteratively, we have

$$\partial_t(P^k u) + H_x\partial_x^2(P^k u) = (P+2)^k Lu = -(P+2)^k\partial_x(u^2). \tag{2.2}$$

Setting $u_k = P^k u$ and $B_k(u, u) = -(P+2)^k\partial_x u^2$, we have

$$\partial_t u_k + H_x\partial_x^2 u_k = B_k(u, u) = -\partial_x \sum_{k=k_0+k_1+k_2} \frac{k!}{k_0!k_1!k_2!} u_{k_1} u_{k_2}. \tag{2.3}$$

An important point here is that the nonlinear terms $B_k(u, u)$ maintain the bilinear structure similar to the original Benjamin-Ono equation. This is due to the fact that the Leibniz law can be applicable for an operation of P. Thus each of u_k satisfies the following system of equations;

$$\begin{cases} \partial_t u_k + H_x\partial_x^2 u_k = B_k(u, u), \qquad t, x \in \mathbb{R}, \\ u_k(0, x) = (x\partial_x)^k\phi(x). \end{cases} \tag{2.4}$$

Firstly we establish the local well-posedness of the solution to the following infinitely coupled system of dispersive equation in a proper Sobolev space:

$$
\begin{cases} \partial_t u_k + H_x \partial_x^2 u_k = B_k(u,u), & t, x \in \mathbb{R}, \\ u_k(0,x) = \phi_k(x). \end{cases} \tag{2.5}
$$

Taking $\phi_k = (x\partial_x)^k \phi(x)$, the uniqueness and local well-posedness allow us to say $u_k = P^k u$ for all $k = 0, 1, \cdots$.

Through showing the existence and uniqueness process, we obtain the estimate

$$
\|P^k u\|_{H^s} \leq C A^k k!.
$$

Until this step, there is no effect from the appearance of the non local operator H_x.

Next we would derive the pointwise derivative estimate by using the equation:

$$
H_x \partial_x^2 P^k u = -\frac{1}{2t} P^{k+1} u + \frac{1}{2t} x \partial_x P^k u + B_k(u,u). \tag{2.6}
$$

To treat the second term of the right hand side of (2.6), we employ localization argument. With a suitable decaying weight function $a = a(x)$, we can show that

$$
\|a \partial_x^l P^k u(t)\|_{H^1(\mathbb{R})} \leq C \langle t^{-1} \rangle^l A^{k+l} (k+l)!, \quad k, l = 0, 1, 2, \cdots.
$$

and then by iterative argument, we can shift from the estimate with the operator P to the one with $t\partial_t$ and conclude

$$
\|(t\partial_t)^{l_1} \partial_x^{l_2} u(t)\|_{L^\infty(x_0-\delta, x_0+\delta)} \leq C \langle t^{-1} \rangle^{l_1+l_2} \langle x_0 \rangle^{3l_1+2l_2} A^{l_1+l_2} (l_1+l_2)!, \tag{2.7}
$$

for $l_1, l_2 = 0, 1, 2, \cdots$. A crucial step for obtaining the above derivative estimates is to treat the nonlocal operator H_x which is an essential difference from the KdV equation or nonlinear Schrödinger equations. In order to handle this term, it is required to show an explicit dependence of the iteration of the commutator estimate

$$
\|[H_x, a^k]\|_{\mathcal{L}(L^2 \to L^2)} \leq C_k,
$$

where $a = a(x)$ is a cut-off function and $a^k = a(x)^k$. We then choose a particular weight function $a(x) = \langle x \rangle^{-2}$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$ and derive an explicit commutation estimate with the Hilbert transform and a^k. By this step, we may use the equation (2.6) to gain the regularity and to show the analyticity (2.7). Here we only exhibit the following lemma which treats the commutator of H_x and a^k.

Lemma 2.1. *If $\|a^l \partial_x^l f\|_2 \leq C A^l l! \|f\|_2$ for $0 \leq l \leq N-1$, then we have*

$$
\|[H_x, a^N] \partial_x^N f\|_2 \leq C A^N N! \|f\|_2.
$$

To prove Lemma 2.1, we use the equality

$$[H_x, a]f = p.v. \int_{\mathbb{R}} \frac{a(y) - a(x)}{x - y} f(y)dy = \int_{\mathbb{R}} \frac{x + y}{\langle x \rangle^2 \langle y \rangle^2} f(y)dy.$$

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