On the Gevrey wellposedness of the Cauchy problem for weakly hyperbolic equations of higher order

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We shall consider the Cauchy problem in $[0,T]\times\mathbf{R}^1_x$

$$(1)\qquad \begin{cases} \partial_t^m u = \displaystyle\sum_{j+k=m,\ j<m} a_{j,k}(t)\partial_t^j\partial_x^k u + \sum_{j+k\le l} a_{j,k}(t)\partial_t^j\partial_x^k u \\ \partial_t^j u(0,x) = u_j(x) \quad (j=0,1,\cdots,m-1), \end{cases}$$

where $0\le l\le m-1$ and $a_{j,k}(t)\in C^0[0,T]$ for j, k satisfying $j+k\le l$.

Now we assume that the characteristic roots $\tau_1(t)\xi,\tau_2(t)\xi,\cdots,\tau_m(t)\xi$ of the characteristic equation $\tau^m=\sum_{j+k=m,\ j<m}a_{j,k}(t)\tau^j\xi^k$ are real valued and satisfy the following conditions

(2) $\tau_1(t),\cdots,\tau_r(t)$ coincide at the same points such that for any $1\le i,p,q\le r(i\ne p,q)$ $|\tau_i(t)-\tau_p(t)|\le {}^\exists C|\tau_i(t)-\tau_q(t)|$ for $t\in[0,T]$ and belong to $C^\alpha[0,T]$ $(0\le\alpha\le 1)$,

(3) $\tau_{r+1}(t),\cdots,\tau_m(t)$ are di stinct and belong to $C^\beta[0,T]$ $(0\le\beta\le 1)$,

where $1\le r\le m$. In particular when $r=1$, $\tau_1(t)$ also becomes a distinct root. Therefore (2) and (3) imply that

(4) $\tau_1(t),\tau_2(t),\cdots,\tau_m(t)$ are distinct and belong to $C^\beta[0,T]$ $(0\le\beta(=\alpha)\le 1)$.

When $m=2$, $l=1$ and $r=2$ (resp. $r=1$), F. Colombini, E. Jannelli, S. Spagnolo and E. De Giorgi assumed $a_{1,1}(t)\equiv 0$ and $a_{0,2}(t)\in C^\gamma[0,T]$ $(\gamma\ge 0)$ which means that the characteristic roots satisfy (2) with $\alpha=\frac{\gamma}{2}$(resp. (4) with $\beta=\gamma$), and showed that the Cauchy problem (1) is wellposed in G^s, provided $1\le s<1+\alpha$ $\big($resp. $1\le s<1+\frac{\beta}{1-\beta}\big)$(see [CDS] and [CJS]).

When $m\ge 2$, $l=m-1$ and $r\ge 2$, Y. Ohya and S. Tarama assumed $a_{j,k}(t)\in C^\gamma[0,T]$ $(0\le$ $gamma$ $\le 2)$ for j, k satisfying $j+k=m$. In their case we remark that the characteristic roots don't always multiply at the same points. Rougly speaking their assumption means that the characteristic roots satisfy (2) with $\alpha=\frac{\gamma}{r}$ and (3) with $\beta=\gamma(=r\alpha)$ from the properties of hyperbolic polynomials(see [B]). Then they showed that the Cauchy problem (1) is wellposed in G^s, provi ded $1\le s<1+\min\{\frac{\gamma}{r},\ \frac{1}{r-1}\}=1+\min\{\alpha,\ \frac{1}{r-1}\}$(see [OT]).

Theorem. *Let T0, $0 \le l \le m-1$ and $2 \le r \le m$(res p. $r=1$). Assume that (2) and (3)(resp. (4)). Then for any $u_j(x) \in G^s(\mathbf{R})(j=0,1,\cdots,m-1)$, the Cauchy problem (1) has a unique solution $u \in C^m\big([0,T]; G^s(\mathbf{R}^1_x)\big)$, provided*

$$1 \le s < 1 + \min\Big\{\alpha,\ \frac{\beta}{r-\beta},\ \ \frac{m-l}{r-1}\Big\}\ \Big(\text{resp. } 1 \le s < 1 + \frac{\beta}{1-\beta}\Big). \tag{5}$$

We shall give the typical example to apply our theorem. We consider the Cauchy problem for the weakly hyperbolic equation of 4th order

$$\begin{cases} \partial_t^4 u = \big\{a(t)+b(t)\big\}\partial_t^2\partial_x^2 u - a(t)b(t)\partial_x^4 u \\ \partial_t^j u(0,x) = u_j(x) \quad (j=0,1,2,3), \end{cases} \tag{6}$$

where $a(t)$ and $b(t)$ belong to $C^{2\alpha}[0,T]$ and $C^{\beta}[0,T]$ respectively and satisfy $a(t) \ge 0$ and $b(t)-a(t) \ge {}^{\exists}\delta 0$ wh ich imply that the multiplicity $r=2$. Since the coefficients belong to $C^{\gamma}[0,T]$ where $\gamma = \min\{2\alpha, \beta\}$, according to [OT] the Cauchy problem (6) is wellposed in G^s, provided

$$1 \le s < 1 + \frac{\gamma}{2} = 1 + \min\Big\{\alpha,\ \frac{\beta}{2}\Big\}.$$

Noting that $b(t)$ is strictly positive, we see that $\tau_1(t) \equiv -\sqrt{a(t)} \in C^{\alpha}[0,T]$, $\tau_2(t) \equiv \sqrt{a(t)} \in C^{\alpha}[0,T]$, $\tau_3(t) \equiv -\sqrt{b(t)} \in C^{\beta}[0,T]$ and $\tau_4(t) \equiv \sqrt{b(t)} \in C^{\beta}[0,T]$. Applying our theorem, we find that the Cauchy problem (7) is well-posed in G^s, provided

$$1 \le s < 1 + \min\Big\{\alpha,\ \frac{\beta}{2-\beta}\Big\}.$$

NOTATIONS

$G^s(\mathbf{R})(s \ge 1)$ is the space of Gevrey functions $f(x)$ satisfying for any compact set $K \subset \mathbf{R}$, $\sup_{x\in K} |D^{\alpha} f(x)| \le C_K \rho_K^{\alpha} \alpha!^s$ for ${}^{\forall}\alpha \in \mathbf{N}$.

REFERENCES

[B] M.D. Bronštein, The Cauchy problem for hyperbolic operators with characteristics of variable multiplicity, *Trudy Moskov. Mat. Obšč.* **41** (1980), 87-103 (Trans. *Moscow Math. Soc.*, **1** (1982), 87-103).

[CDS] F. Colombini, E. De Giorgi and S. Spagnolo, Sur les é quations hyperboliques avec des coefficients qui ne dépendent que du temps, *Ann. Scuola Norm Sup. Pisa*, **6** (1979), 511-559.

[CJS] F. Colombini, E. Jannelli and S. Spagnolo, Wellposedness in the Gevrey classes of the Cauchy problem for a non strictly hyperbolic equation with coefficients depending on time, *Ann. Scuola Norm Sup. Pisa*, **10** (1983), 291-312.

[KWY] K. Kajitani, S. Wakabayashi and K. Yagdjian, The C^{∞}-wellposed Cauchy problem for hyperbolic operators with multiple characteristics vanishing with the different speeds, preprint.

[OT] Y. Ohya and S. Tarama, Le problème de Cauchy à caractéristiques multiples -coefficients hölderiens en t- , (*Proc. Taniguchi Intern. Sympos. on Hyperbolic Equations and Related Topics* 1984), *Kinokuniya*, 1986, 273-306.