GENERALIZED FUNCTIONS AND OPERATIONAL CALCULUS DISCUSSED BY FOURIER AND HEAVISIDE

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Schwartz' book [9] on distributions starts with quotations of Heaviside's paper [3] on operational calculus and Dirac's [1] on quantum mechanics in which Dirac introduced the δ-function. These are supposed to be the first places where distributions were actually employed.

Little is known, however, the fact that Fourier defined much earlier the δ-function as an integral kernel, as Dirac, and formulated his integral formula as

$$\delta(x-\alpha) = \frac{1}{\pi}\int_0^\infty \cos(q(x-\alpha))dq. \tag{1}$$

His book [2] claims this repeatedly. In particular, the statement on page 449 is incontestable. Moreover, his proof of (1) given on pages 546–551 is the same as that of Dirichlet, Riemann and Jordan and is sufficient for developing his "fonctions quelconques" which are actually piecewise real-analytic functions.

Fourier did not know the ϵ-δ arguments. Therefore, it is easy to find fault with his proof. However, reading Riemann's papers [7], [8] etc., we find that our concept of arbitrary functions was given birth to by Fourier's integral representation of arbitrary functions, and that accordingly one had to redefine the integral for such a function. The δ-function might have been easier for one to understand when the infinitesimal and the infinity existed in reality.

On pp. 511–546 of [2] Fourier solves not only the heat equation but also other partial differential equations by means of the Fourier transformation (and erroneously by the power series expansion). For example, he obtained the solution of the initial value problem of the wave equation in two dimensions as

$$u(t,x,y) = (\cos(t\sqrt{-\Delta})\phi)(x,y) + \int_0^t (\cos(s\sqrt{-\Delta})\psi)(x,y)ds. \tag{2}$$

Solutions to linear ordinary differential equations with analytic data are analytic except at singular points of equations and data, and solutions for nonanalytic data can be approximated by analytic solutions for analytic data. Fourier seems to have thought that this was also the case with partial differential equations. For example, he claimed that the solution of the Laplace equation was represented as (2) with $-\Delta$ replaced by Δ with two initial values. The right representation is, of course, the Poisson integral $(e^{-t\sqrt{-\Delta}}\phi)(x,y)$ with one data.

More serious is his error with the boundary value problem of the heat equation on the half line $\{x \geq 0\}$. When a boundary value $\phi(t)$ at $\{x = 0\}$ is given, the most practical solution is represented as $(e^{-x\sqrt{\partial/\partial t}}\phi)(t)$ but he made it $(\cos(x\sqrt{-\partial/\partial t})\phi)(t) + \int_0^t (\cos(x\sqrt{-\partial/\partial t})\psi)(s)ds$ with two data. Later Fourier corrected this error. When W. Thomson applied the result to the theory of cables for telegram [10], he quoted the right representation. Heaviside's theory of operational calculus [3], [4] was invented first to make the Fourier-Thomson theory more accessible to engineers and then to establish the theory of cables for telephone. Dirac obtained his ideas from Heaviside's theory.

Fourier's results mentioned above are all stated in the last Chapter IX, pp. 425–601, [2]. Few people reach there because it requires a great patience to read through the first parts of the book. It is a collection of many papers written at diverse times and is not edited well. Gauss' formula was not yet known, which makes the derivation of the heat equation difficult, etc.

That does not mean at all that the first parts are not interesting. For example, the first Fourier series he obtained is

$$\operatorname{sign} \cos y = \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\cos(2n-1)y}{2n-1}, \tag{3}$$

of which he gave seven proofs in the book. The fourth proof on pp. 189–190 is comprehensible only if we regard the left-hand side as a hyperfunction.

After we have an operational representation like (2), we have to carry out difficult integration in order to obtain the solution $u(t, x, y)$ or the integral kernel for the resolvent $\cos(t\sqrt{-\Delta})$. Heaviside employed instead the fractional power series expansions in operator $p = d/dt$ and computed the integrals as series of distributions with the use of the following correspondence of operators and distributions:

$$p^\alpha \longleftrightarrow \begin{cases} \dfrac{t_+^{-\alpha-1}}{\Gamma(-\alpha)}, & \alpha \neq 0, 1, 2, \cdots, \\ \delta^{(\alpha)}(t), & \alpha = 0, 1, 2, \cdots. \end{cases} \tag{4}$$

Even when the obtained expansion diverges, it often represents an asymptotic expansion.

This is his method which Schwartz [9] commented as 'audacious'. It was more so at the end of the nineteenth century. Heaviside's paper [3] originally consisted of three parts. After two first parts were published, the third one was rejected as nonsense and was never published (cf. [4]). Today we can righteously interpret most of his results by means of the theory of hyperfunctions and ultradistributions [5], [6].

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