A SMOOTHING PROPERTY OF SCHRÖDINGER EQUATIONS ALONG THE SPHERE

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We shall consider the following Cauchy problem of the Schrödinger equation

$$\begin{cases} (i\partial_t + \triangle_x)u(t,x) = 0 \\ \qquad\qquad u(0,x) = \varphi(x), \end{cases} \tag{1}$$

where $u = u(t,x) \in \mathcal{S}'(\mathbf{R}_t \times \mathbf{R}^n_x)$ and $\varphi(x) \in \mathcal{S}(\mathbf{R}^n)$. Since the solution operator $e^{it\triangle}$ conserves the L^2-norm, the solution $u(t,x)$ can be defined for $\varphi(x) \in L^2(\mathbf{R}^n)$, as well, and belongs to $L^2(\mathbf{R}^n_x)$ for any fixed t. But, by integrating the solution in t, the extra gain of regularity in x can be observed. For example, in the case $n = 1$, we have the inequality

$$\sup_{x\in\mathbf{R}} \left\| |D_x|^{1/2} u(\cdot, x) \right\|_{L^2(\mathbf{R}_t)} \le \|\varphi\|_{L^2(\mathbf{R})}$$

by an appropriate use of Plancherel's theorem. In the higher dimensional case $n \ge 2$, Sjölin [4] shows that $\langle D_x\rangle^{1/2} u \in L^2_{loc}(\mathbf{R}_t \times \mathbf{R}^n_x)$ for $\varphi(x) \in L^2(\mathbf{R}^n)$.

On the other hand, Kato and Yajima [3] showed their global version, that is, the inequality

$$\left\| |x|^{\alpha-1} |D_x|^{\alpha} u \right\|_{L^2(\mathbf{R}_t\times\mathbf{R}^n_x)} \le C \|\varphi\|_{L^2(\mathbf{R}^n)}, \tag{2}$$

where $0 \le \alpha < 1/2$ in the case $n \ge 3$ and $0 < \alpha < 1/2$ in the case $n = 2$. This result has been extended to the case $1 - n/2 < \alpha < 1/2$ in the case $n \ge 2$ by Sugimoto [5].

As for the problem whether we can obtain the inequality (2) with $\alpha = 1/2$, Hoshiro [2] proved, in the case $n \ge 3$ and $0 \le \alpha < 1/2$,

$$\left\| |x|^{\alpha-1} \Omega^{1/2-\alpha} |D_x|^{\alpha} u \right\|_{L^2(\mathbf{R}_t\times\mathbf{R}^n_x)} \le C \|\varphi\|_{L^2(\mathbf{R}^n)}. \tag{3}$$

Here Ω^σ denotes the homogeneous extension of the elliptic self-adjoint operator $(1 - \triangle_{S^{n-1}})^{\sigma/2}$ on the sphere S^{n-1}, and $\triangle_{S^{n-1}}$ denotes the Laplace-Beltrami operator on S^{n-1}. For example, for $k \in 2\mathbf{N}$, we have

$$\Omega^k = \left(1 - \sum_{i<j} \left(x_j \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_i} \right)^2 \right)^{k/2}.$$

If we remark that Ω^σ behaves like $|x|^\sigma |D_x|^\sigma$ in the sense of the orders of differential and decay, the estimate (3) means that the gain of regularity is exactly the order $1/2$ just like the local smoothing property proved by Sjölin [4].

The objective of this article is to investigate why the sphere S^{n-1} has such a special meaning. For the purpose, we shall replace the Laplacian $-\triangle_x$ in (1) by more general elliptic operators $L_p = p(D)^2$, where $p(\xi)$ is the square root of a positive definite quadratic form. The case $p(\xi) = |\xi|$ corresponds to the case $L_p = -\triangle_x$. We shall give an answer to the question by finding the surface which has the same property in this generalized case.

Our main theorem below says that the surface

$$\Sigma_p' = \{\nabla p(\xi); p(\xi) = 1\}$$

plays the same role as the sphere S^{n-1} does in the special case $p(\xi) = |\xi|$. We can replace $\Omega^{1/2-\alpha}$ in the estimate (3) by $\Omega_{p'}^{1/2-\alpha}$, which is defined by using the Laplace-Beltrami operator on Σ_p' instead of $\triangle_{S^{n-1}}$. Furthermore, it says that the estimate (3) can be extended to the case $n \geq 2$ and $1 - n/2 < \alpha < 1/2$.

Theorem 1. Suppose $n \geq 2$ and $1 - n/2 < \alpha < 1/2$. Then the solution $u(t,x) \in \mathcal{S}'(\mathbf{R}_t \times \mathbf{R}_x^n)$ to the problem

$$\begin{cases} (i\partial_t - L_p)u(t,x) = 0 \\ \qquad\qquad u(0,x) = \varphi(x) \in \mathcal{S}(\mathbf{R}_x^n) \end{cases}$$

satisfies

$$\left\| |x|^{\alpha-1} \Omega_{p'}^{1/2-\alpha} |p(D)|^{\alpha} u(t,x) \right\|_{L^2(\mathbf{R}_t \times \mathbf{R}_x^n)} \leq C \|\varphi\|_{L^2(\mathbf{R}_x^n)}.$$

The surface Σ_p' is the set of the positions of particles at the time $t = 1$, which started from the origin at $t = 0$ and traveled along the classical orbits. This theorem suggests that the classical orbit propagates some information on unknown regularity properties.

We remark that Theorem 1 cannot be obtained from a simple modification of the proof of the estimate (3). Since the method in [2] depends on the special function theory, it can be applied to only the special case $L_p = -\triangle_x$. The change of variable does not directly imply our theorem either.

The essential part of the proof is to show a refinement of the limiting absorption principle, the fact that the resolvent $(L_p - \mu)^{-1}$ has a meaning even if μ is a spectrum. This principle was first shown by Agmon [1], which says that $(L_p - \mu)^{-1}$ has the weak limit as a bounded operator on a weighted L^2-space of the order greater than $1/2$, when μ tends to the spectrum. More precisely, for each $d \in \mathbf{R}$, we have

$$\sup_{\varepsilon>0} \left\| (1+|x|)^{-k} \left(L_p - (d+i\varepsilon)^2\right)^{-1} v \right\|_{L^2(\mathbf{R}^n)} \leq C_d \left\| (1+|x|)^l v \right\|_{L^2(\mathbf{R}^n)}$$

if $k > 1/2$ and $l > 1/2$. To prove Theorem 1, we need the following estimate which says that we can attain the critical order $1/2$ for the weight if we consider an effect along the surface $\Sigma_{p'}$:

Theorem 2. Let $d \in \mathbf{R}$ and $\Psi \in C_0^\infty(\mathbf{R}^n)$. Suppose $k, l > 1/2$. Then we have

$$\begin{aligned} &\sup_{\varepsilon>0} \left\| (1+|x|)^{-k} \Omega_{p'}^{k-1/2} \left(L_p - (d+i\varepsilon)^2\right)^{-1} \Psi(D) v \right\|_{L^2(\mathbf{R}^n)} \\ &\leq C_{d,\Psi} \left\| (1+|x|)^l \left(\Omega_{p'}^{1/2-l}\right)^* v \right\|_{L^2(\mathbf{R}^n)}. \end{aligned}$$

References

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