WELL-POSEDNESS FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH DERIVATIVE NONLINEARITY

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This note concerns the well-posedness for the Cauchy problem of derivative nonlinear Schrödinger equation:

$$i\partial_t u + \partial_x^2 u = i\lambda\partial_x(|u|^2 u), \quad (t,x) \in \mathbb{R}^2, \tag{1}$$

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}, \tag{2}$$

where the unknown function u is a complex valued function of $(t,x) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. The derivative nonlinear Schrödinger equation is a model of the propagation of circularly polarized Alfvèn waves in magnetized plasma with a constant external magnetic field [15, 16].

Here we consider the well-posedness, which means the existence, the uniqueness, the persistency property ($u_0 \in X \Rightarrow u(t) \in X$) and the continuous dependence of solution on data ($u_0 \to v_0 \Rightarrow u(t) \to v(t)$). We say the global (resp. local) well-posedness, if the well-posedness holds for all time (resp. a positive time interval).

When we try to show the well-posedness by the contraction argument, we meet with the derivative loss stemming from the derivative nonlinearity of equation (1). In [9, 10, 11, 17], the H^1 global well-posedness for small initial data was shown, where the proof uses two gauge transformations. The strategy of argument in [9, 10, 11, 17] is to reduce the original equation (1) to equations without derivative nonlinearity.

The well-posedness below energy is considered in [18, 20]. In [18] the $H^{\frac{1}{2}}$ local well-posedness was proven. The proof of [18] was a suitable combination of the gauge transformation and the Fourier restriction norm method [1, 12]. The transformation [9, 10, 11, 17, 18]

$$v(t,x) = \exp\left(-i\lambda\int_{-\infty}^{x} |u(t,y)|^2 dy\right) u(t,x), \tag{3}$$

reduces (1) to the following:

$$i\partial_t v + \partial_x^2 v = -i\lambda v^2 \partial_x \overline{v} - \frac{\lambda^2}{2}|v|^4 v. \tag{4}$$

Solutions of the Cauchy problem corresponding to (5) satisfy the conservation law:

$$(5) \qquad E_1(v(t)) = ||\partial_x v(t)||^2_{L^2_x} - \frac{\lambda}{2}\Im\langle |v(t)|^2 v(t), \partial_x v(t)\rangle_{L^2_x}.$$

As compared to the equation (1), $|v|^4 v$ appears and $v^2\partial_x \overline{v}$ remains in (3), but the term of derivative type $|v|^2\partial_x v$ in (1) is removed here. The term $|v|^2\partial_x v$ is difficult to handle, while the proof in [18] can control the terms $v^2\partial_x\overline{v}$ and $|v|^4 v$.

Anyway, it is natural to try to extend the above local solution below energy to global one. But it is not obvious, because it seems difficult to give the a priori bound of solution by the usual use of conservation laws (cf. (5), L^2-conservation, etc ...). The above problem is investigated in [20]. The proof of [20] is based on the argument in [3]. In [3] Bourgain shows global existence results for the 2D-NLS in weaker spaces than in the energy space (see also [4, 5, 3D-NLS], [6, 7, KdV], [8, mKdV], [13, 14, NLW], [19, 23, KP-2]). His strategy is a general scheme as follows. Let V_t and $V(t)$ denote the flow maps of nonlinear and linear equations, that is, $V_t u_0$ and $V(t)u_0$ are solutions of nonlinear and linear equations, respectively. Let X and Y be Banach spaces such that $X \subsetneq Y$, where X and Y correspond to conservation law spaces and the initial data space, respectively. His strategy is that if $(V_t - V(t))u_0 \in X$ for all time t, we have the global well-posedness in Y. Roughly speaking, his argument aims to estimate the high Sobolev norm of solution by the low Sobolev norm, which controls the transportation of energy from the low frequency region to the high frequency region. He applied this argument to NLS with $X = H^1$ and $Y = H^s$ for some $0 < s < 1$ in [3]. However in equations of derivative type, the derivative loss stems from the derivative nonlinearity. Moreover we have to regain more than one derivatives for the usage of Bourgain's argument. Here we combine the above argument of Bourgain with the Fourier restriction norm method observed in [18].

This talk devotes to the above results of [18, 20]. More precisely, we are to exhibit the following two theorems.

Theorem 1 ([18]). *Let $s \geq \frac{1}{2}$. Then the Cauchy problem (1)-(2) is locally well-posed in H^s.*

Theorem 2 ([20]). *The Cauchy problem (1)-(2) is globally well-posed in H^s for $s_0 < s < 1$ for appropriate $s_0 \in (\frac{1}{2}, 1)$, provided $||u_0||_{L^2} < \sqrt{\frac{2\pi}{|\lambda|}}$.*

Concerning the number s_0 in Theorem 2, we shall observe in the talk.

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