# Operator inequalities and their applications to non-normal operators

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## 1 Order preserving operator inequalities

In this report, an operator means a bounded linear operator on a Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$, and also $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

We begin this report by introducing the following result which is quite useful for the study of the class of operators including normal operators ($\Longleftrightarrow T^*T = TT^*$).

**Theorem F (Furuta inequality [18]).**
*If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $\quad (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$

*and*

(ii) $\quad (A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$

*hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p + r$.*

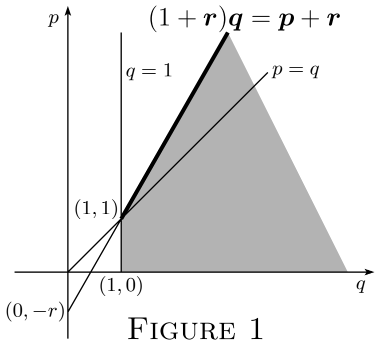



FIGURE 1

We remark that Theorem F yields Löwner-Heinz theorem "$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$" when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem F are given in [10][35] and also an elementary one-page proof in [19]. It is shown in [37] that the domain drawn for $p, q$ and $r$ in Figure 1 is the best possible for Theorem F.

## 2 $p$-Hyponormal and log-hyponormal operators

The following are well-known classes of non-normal operators.

**Definition.**

(i) $T$ : hyponormal $\Longleftrightarrow T^*T \geq TT^*$.

(ii) $T$ : $p$-hyponormal for $p > 0 \Longleftrightarrow (T^*T)^p \geq (TT^*)^p$.

(iii) $T$ : log-hyponormal $\Longleftrightarrow T$ is invertible and $\log T^*T \geq \log TT^*$.

(iv) $T$ : paranormal $\Longleftrightarrow \|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$.

It is well known that the following inclusion relations among these classes hold.

**Proposition.**

(i) $T$ : *hyponormal* $\Longleftrightarrow T$ : 1-*hyponormal.*

(ii) $T$ : *p-hyponormal* $\Longrightarrow T$ : *q-hyponormal for any* $p \geq q > 0$.

(iii) $T$ : *invertible and p-hyponormal for some* $p > 0 \Longrightarrow T$ : *log-hyponormal.*

(iv) $T$ : *p-hyponormal for some* $p > 0$ *or log-hyponormal* $\Longrightarrow T$ : *paranormal* ([6][24]).

(ii) follows from Löwner-Heinz theorem, and (iii) holds since $\log t$ is operator monotone. We remark that $p$-hyponormality tends to log-hyponormality as $p \to +0$ since $\frac{X^p - I}{p} \to \log X$ as $p \to +0$ for every positive invertible operator $X$. (iv) shall be discussed in §7.

It is also well known that all of the inclusion relations shown in Proposition are proper. Especially, it can be proved by the following example about (ii) and (iii).

**Lemma 2.1 ([49]).** *For positive operators $A$ and $B$ on $H$, define the operator $T$ on $\bigoplus_{k=-\infty}^{\infty} H$ as follows:*

$$T = \begin{pmatrix} \ddots & & & & & & & \\ \ddots & 0 & & & & & \mathbf{0} & \\ & B^{\frac{1}{2}} & 0 & & & & & \\ & & B^{\frac{1}{2}} & \boxed{0} & & & & \\ & & & A^{\frac{1}{2}} & 0 & & & \\ & \mathbf{0} & & & A^{\frac{1}{2}} & 0 & & \\ & & & & & \ddots & \ddots & \end{pmatrix}, \tag{2.1}$$

*where* $\square$ *shows the place of the* $(0,0)$ *matrix element. Then*

(i) $T$ : *p-hyponormal for* $p > 0 \Longleftrightarrow A^p \geq B^p$.

(ii) $T$ : *log-hyponormal* $\Longleftrightarrow A$ *and* $B$ *are invertible and* $\log A \geq \log B$.

Associated with Löwner-Heinz theorem, it is well known that $A \geq B \geq 0$ does not always ensure $A^{\alpha} \ngeq B^{\alpha}$ for $\alpha > 1$. Put

$$A_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

for example, then $A_0 \geq B_0 \geq 0$ and $A_0^{\alpha} \ngeq B_0^{\alpha}$ for all $\alpha > 1$ ([23]). For each $q > p > 0$, put $A = A_0^{\frac{1}{p}}$ and $B = B_0^{\frac{1}{p}}$, and define $T$ as (2.1), then it turns out by (i) of Lemma 2.1 that $T$ is $p$-hyponormal and not $q$-hyponormal. Next, put $A, B > 0$ as

$$\log A = \begin{pmatrix} 2 & \sqrt{6} \\ \sqrt{6} & 1 \end{pmatrix} \quad \text{and} \quad \log B = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix},$$

then $\log A \geq \log B$ and $A^p \ngeq B^p$ for all $p > 0$ ([15]). Define $T$ as (2.1), then it turns out by (ii) of Lemma 2.1 that $T$ is log-hyponormal and not $p$-hyponormal for all $p > 0$.

# 3 Aluthge transformation of $p$-hyponormal and log-hyponormal operators

The operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is called Aluthge transformation of an operator $T$ whose polar decomposition is $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$. It was shown in [4][8] that $\sigma(\tilde{T}) = \sigma(T)$ holds for any operator $T$, where $\sigma(T)$ denotes the spectrum of $T$. Aluthge transformation was first introduced by Aluthge [1], and he showed the following result on Aluthge transformation of $p$-hyponormal operators.

**Theorem 3.A ([1]).** *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator for $0 < p < 1$ and $U$ be unitary. Then*

(i) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is* $(p+\frac{1}{2})$*-hyponormal if* $0 < p \leq \frac{1}{2}$.

(ii) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is hyponormal if* $\frac{1}{2} \leq p < 1$.

Huruya [29] and Yoshino [50] showed the following result which is an extension of Theorem 3.A on generalized Aluthge transformation $\tilde{T}_{s,t} = |T|^sU|T|^t$ of $p$-hyponormal operators.

**Theorem 3.B ([29][50]).** *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator for $p > 0$. Then the following assertions hold for $s > 0$ and $t > 0$:*

(i) $\tilde{T}_{s,t} = |T|^sU|T|^t$ *is* $\frac{p+\min\{s,t\}}{s+t}$*-hyponormal if* $\max\{s,t\} \geq p$.

(ii) $\tilde{T}_{s,t} = |T|^sU|T|^t$ *is hyponormal if* $\max\{s,t\} \leq p$.

Tanahashi [38] showed a parallel result to Theorem 3.B for log-hyponormal operators as an application of Theorem 4.A in the following section.

**Theorem 3.C ([38]).** *Let $T = U|T|$ be the polar decomposition of a log-hyponormal operator. Then $\tilde{T}_{s,t} = |T|^sU|T|^t$ is $\frac{\min\{s,t\}}{s+t}$-hyponormal for any $s > 0$ and $t > 0$.*

# 4 A characterization of the chaotic order and its best possibility

The order between positive invertible operators $A$ and $B$ defined by $\log A \geq \log B$ is called chaotic order. The chaotic order is weaker than the usual order $A \geq B$ since $\log t$ is operator monotone. The following result is a characterization of the chaotic order of Theorem F type, which is an extension of a result by Ando [7]. Among others, a simple proof was given by Uchiyama [39] recently.

**Theorem 4.A ([11][13][21][39]).** *For positive invertible operators $A$ and $B$, $\log A \geq \log B$ if and only if*

$$A^r \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$$

*holds for all $p \geq 0$ and $r \geq 0$.*

Theorem 4.A can be rewritten as follows.

**Theorem 4.A'.** *For positive and invertible operators $A$ and $B$, $\log A \geq \log B$ if and only if*

$$A^{\frac{p+r}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

*holds for all $p \geq 0$ and $r \geq 0$ with $rq \geq p+r$.*

We consider an order between positive operators $A$ and $B$ defined by $A^\delta \geq B^\delta$ for a fixed $\delta > 0$. It is obvious by Löwner-Heinz theorem that $A^{\delta_1} \geq B^{\delta_1}$ implies $A^{\delta_2} \geq B^{\delta_2}$ for $\delta_1 \geq \delta_2 > 0$, so that the order $A^\delta \geq B^\delta$ interpolates between the usual and chaotic orders continuously since it coincides with the usual order $A \geq B$ in case $\delta = 1$, and tends to the chaotic order $\log A \geq \log B$ as $\delta \to +0$ since $\frac{X^p - I}{p} \to \log X$ as $p \to +0$ for every positive invertible operator $X$. The following result can be obtained by applying Theorem F to $A^\delta \geq B^\delta \geq 0$.

**Theorem 4.B ([14][15]).** *For positive operators $A$ and $B$, $A^\delta \geq B^\delta$ for a fixed $\delta > 0$ if and only if*

$$A^{\frac{p+r}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

*holds for all $p \geq 0$, $r \geq 0$ and $q \geq 1$ with $(\delta + r)q \geq p+r$.*

Figure 2 [15] shows the domains of parameters $p$, $q$ and $r$ where the inequality $A^{\frac{p+r}{q}} \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$ holds under the conditions $A \geq B$, $A^\delta \geq B^\delta$ for $\delta \in (0,1)$ and $\log A \geq \log B$ by Theorem F, Theorem 4.B and Theorem 4.A'. It turns out that the domains where the inequality holds are smaller as the orders in the assumption are weaker.

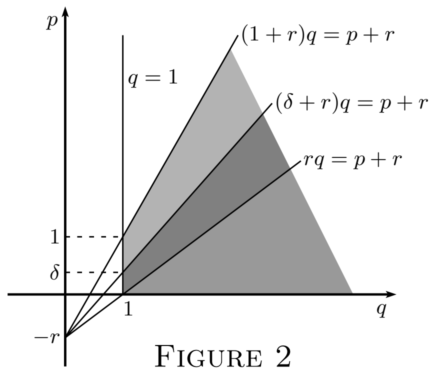



FIGURE 2

Tanahashi [37] showed the following result which states that the domain of parameters of Theorem F for the usual order is the best possible.

**Theorem 4.C ([37]).** *Let $p > 0$, $q > 0$ and $r > 0$. If $0 < q < 1$ or $(1+r)q < p+r$, there exist $A, B > 0$ on $\mathbb{R}^2$ such that $A \geq B > 0$ and*

$$A^{\frac{p+r}{q}} \not\geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}.$$

By applying Theorem 4.C and using the idea of Figure 2, we showed the following result which states that the domain of the parameters of Theorem 4.A for the chaotic order is also the best possible.

**Theorem 4.1 ([49]).** *Let $p > 0$ and $r > 0$. If $\alpha > 1$, then there exist $A, B > 0$ on $\mathbb{R}^2$ such that $\log A \geq \log B$ and*

$$A^{r\alpha} \not\geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r\alpha}{p+r}}.$$

# 5 The best possibilities of the results on Aluthge transformation

As applications of Theorem 4.C and Theorem 4.1 we also show the following results which state the best possibilities of Theorem 3.B and Theorem 3.C, respectively.

**Theorem 5.1 ([49]).** *Let $p > 0$, $s > 0$ and $t > 0$. And let $T = U|T|$ be the polar decomposition of $T$. Then the following assertions hold:*

(i) *In case $\max\{s, t\} \geq p$, if $\alpha > \frac{p+\min\{s,t\}}{s+t}$, there exists a $p$-hyponormal operator $T$ such that $\tilde{T}_{s,t} = |T|^s U|T|^t$ is not $\alpha$-hyponormal.*

(ii) *In case $p \geq \max\{s, t\}$, if $\alpha > 1$, there exists a $p$-hyponormal operator $T$ such that $\tilde{T}_{s,t} = |T|^s U|T|^t$ is not $\alpha$-hyponormal.*

**Theorem 5.2 ([49]).** *Let $s > 0$ and $t > 0$. If $\alpha > \frac{\min\{s,t\}}{s+t}$, there exists a log-hyponormal operator $T$ such that $\tilde{T}_{s,t} = |T|^s U|T|^t$ is not $\alpha$-hyponormal, where $T = U|T|$ is the polar decomposition of $T$.*

Especially, the following result can be obtained as a simple corollary of Theorem 5.1.

**Corollary 5.3.** *For each $\varepsilon > 0$, there exists a hyponormal operator $T$ such that $\tilde{T} = |T|^{\frac{1}{2}} U|T|^{\frac{1}{2}}$ is not $(1 + \varepsilon)$-hyponormal.*

The following problems remain for the Aluthge transformation $\tilde{T}$ and the $n$-th Aluthge transformation $\tilde{T}^{(n)}$ whose definition is introduced in the following section.

**Conjecture 5.A ([34]).** *There exists a hyponormal operator $T$ such that $\tilde{T}$ is not $(1 + \varepsilon)$-hyponormal for every $\varepsilon > 0$.*

**Conjecture 5.4.** *There exists a hyponormal operator $T$ such that $\tilde{T}^{(n)}$ is not $(1 + \varepsilon)$-hyponormal for every $\varepsilon > 0$ and positive integer $n$.*

# 6 Repeated Aluthge transformation

In this section, we shall introduce recent results by Yamazaki [42][43][44][45] on repeated Aluthge transformation defined as follows.

**Definition ([34][42]).** The $n$-th Aluthge transformation $\tilde{T}^{(n)}$ of an operator $T$ is defined by $\tilde{T}^{(1)} = \tilde{T}$ and $\tilde{T}^{(n+1)} = \widetilde{\tilde{T}^{(n)}}$ for every positive integer $n$.

Firstly, we shall introduce results on norms of repeated Aluthge transformation.

**Theorem 6.A ([42]).** *For every operator $T$ and positive integer $n$, $\|T\| = \|T^{n+1}\|^{\frac{1}{n+1}}$ if and only if $\|T\| = \|\tilde{T}^{(n)}\|$.*

An operator $T$ is said to be normaloid if $\|T\| = r(T)$ where $r(T)$ denotes the spectral radius of $T$. It is well known that $T$ is normaloid if and only if $\|T\| = \|T^n\|^{\frac{1}{n}}$ for all positive integer $n$, so that the following another characterization of normaloid operators can be obtained as an immediate corollary of Theorem 6.A, which is an extension of a result in [12].

**Corollary 6.B ([42]).** *An operator $T$ is normaloid if and only if $\|T\| = \|\tilde{T}^{(n)}\|$ for all positive integer $n$.*

The following can be considered parallel to the well-known fact that $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T)$ holds for any operator $T$.

**Theorem 6.C ([45]).** $\lim_{n\to\infty} \|\tilde{T}^{(n)}\| = r(T)$ *holds for any operator $T$.*

Secondly, we shall introduce results on numerical ranges of repeated Aluthge transformation. Jung-Ko-Pearcy [34] showed the following result and conjecture, where $W(T)$ denotes the numerical range of an operator $T$.

**Theorem 6.D ([34]).** $W(T) \supseteq W(\tilde{T})$ *holds for any $2 \times 2$ matrix $T$.*

**Conjecture 6.E ([34]).** $W(T) \supseteq W(\tilde{T})$ *holds for any operator $T$.*

The following result was shown in [44] related to Theorem 6.D and Conjecture 6.E, where $w(T)$ denotes the numerical radius of an operator $T$.

**Theorem 6.F ([44]).** $w(T) \geq w(\tilde{T})$ *holds for any operator $T$.*

Furthermore, the following results were also shown.

**Theorem 6.G ([44]).** *If $T = U|T|$ for an isometry $U$, then $\overline{W(T)} \supseteq \overline{W(\tilde{T})}$ holds.*

**Corollary 6.H ([44]).** *If $N(T) \subseteq N(T^*)$, then*

$$\overline{W(T)} \supseteq \overline{W(\tilde{T})} \supseteq \overline{W(\tilde{T}^{(2)})} \supseteq \cdots \supseteq \overline{W(\tilde{T}^{(n)})}$$

*holds for every positive integer $n$.*

Theorem 6.G is an extension of Theorem 6.D since $T = U|T|$ for a unitary $U$ and $W(T)$ is a closed set in case $T$ is a matrix.

The following result is important to give proofs of the results in this section.

**Theorem 6.I ([28]).** *Let $A$ and $B$ be positive operators and $X$ be an operator. Then the following assertions hold:*

(i) $\|AXB\|^r \|X\|^{1-r} \geq \|A^r X B^r\|$ *for* $r \in [0, 1]$.

(ii) $\|A^r X B^r\| \|X\|^{r-1} \geq \|AXB\|^r$ *for* $r \geq 1$.

# 7 Powers of $p$-hyponormal and log-hyponormal operators

It is well known that even if an operator $T$ is hyponormal, $T^2$ is not hyponormal in general, but paranormal since every hyponormal operator is paranormal [6] and if $T$ is paranormal, then $T^n$ is also paranormal for any natural number $n$ [17]. Such an example is shown in [27, Problem 209]. Associated with this fact, Aluthge-Wang [3] showed the following result.

**Theorem 7.A ([3]).** *Let $T$ be a $p$-hyponormal operator for $p \in (0,1]$. Then*

$$(T^{n*}T^n)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n*})^{\frac{p}{n}}$$

*holds for all positive integer $n$, i.e., $T^n$ is $\frac{p}{n}$-hyponormal.*

It follows from Theorem 7.A that $T^2$ is $\frac{1}{2}$-hyponormal for every hyponormal operator $T$, which is more precise than the fact mentioned above since every $\frac{1}{2}$-hyponormal operator is paranormal [6][24].

Furuta and the author [25][26] obtained extensions of Theorem 7.A, and Ito [32] obtained the following further extension for $p > 0$.

**Theorem 7.B ([32]).** *Let $T$ be a $p$-hyponormal operator for $p > 0$ and $m$ be a positive integer such that $p \in (m-1, m]$. Then the following assertions hold:*

(i) *$T^{n*}T^n \geq (T^*T)^n$ and $(TT^*)^n \geq T^nT^{n*}$ hold for all positive integer $n < p+1$.*

(ii) *The following inequalities hold for all positive integer $n \geq p+1$:*

$$(T^{n*}T^n)^{\frac{p+1}{n}} \geq \cdots \geq (T^{m+2*}T^{m+2})^{\frac{p+1}{m+2}} \geq (T^{m+1*}T^{m+1})^{\frac{p+1}{m+1}} \geq (T^*T)^{p+1}$$

*and*

$$(TT^*)^{p+1} \geq (T^{m+1}T^{m+1*})^{\frac{p+1}{m+1}} \geq (T^{m+2}T^{m+2*})^{\frac{p+1}{m+2}} \geq \cdots \geq (T^nT^{n*})^{\frac{p+1}{n}}.$$

**Corollary 7.C ([32]).** *Let $T$ be a $p$-hyponormal operator for $p > 0$. Then the following assertions hold:*

(i) *$T^{n*}T^n \geq (T^*T)^n \geq (TT^*)^n \geq T^nT^{n*}$ holds for all positive integer $n < p$, i.e., $T^n$ is hyponormal.*

(ii) *$(T^{n*}T^n)^{\frac{p}{n}} \geq (T^*T)^p \geq (TT^*)^p \geq (T^nT^{n*})^{\frac{p}{n}}$ holds for all positive integer $n \geq p$, i.e., $T^n$ is $\frac{p}{n}$-hyponormal.*

Yamazaki [41] showed the following Theorem 7.D and Corollary 7.E for log-hyponormal operators which correspond to Theorem 7.B and Corollary 7.C in case $p \to 0$, respectively since log-hyponormal can be regarded as 0-hyponormal as mentioned before.

**Theorem 7.D ([41]).** *Let $T$ be a log-hyponormal operator. Then*

$$(T^{n*}T^n)^{\frac{1}{n}} \geq \cdots \geq (T^{2*}T^2)^{\frac{1}{2}} \geq T^*T$$

*and*

$$TT^* \geq (T^2T^{2*})^{\frac{1}{2}} \geq \cdots \geq (T^nT^{n*})^{\frac{1}{n}}$$

*hold for all positive integer $n$.*

**Corollary 7.E ([41]).** *Let $T$ be a log-hyponormal operator. Then*

$$\log(T^{n*}T^n)^{\frac{1}{n}} \geq \log T^*T \geq \log TT^* \geq \log(T^nT^{n*})^{\frac{1}{n}}$$

*holds for all positive integer $n$, i.e., $T^n$ is also log-hyponormal.*

Corollary 7.E yields the following result by Aluthge-Wang [2] "*If $T$ is log-hyponormal, then $T^{2^n}$ is also log-hyponormal for any positive integer $n$.*"

The following result is important to give a proof of Theorem 7.B and Theorem 7.D.

**Theorem F$_f$ ([20][21]).** *If $A \geq B \geq 0$ or $\log A \geq \log B$, then the following assertions hold:*

(i) *For each $q \geq 0$ and $r \geq 0$, $f(p) = (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{q+r}{p+r}}$ is increasing for $p \geq q$.*

(ii) *For each $q \geq 0$ and $r \geq 0$, $g(p) = (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{q+r}{p+r}}$ is decreasing for $p \geq q$.*

# 8 Classes of operators associated with operator inequalities

Furuta-Ito-Yamazaki [24] introduced the following class of operators called class A, and showed the following result on inclusion relations which gives another proof of the result by Ando [6]. We remark that class A is defined by an operator inequality parallel to the norm inequality "$\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$" which defines paranormal operators.

**Definition ([24]).** $T \in$ class A $\Longleftrightarrow |T^2| \geq |T|^2$.

**Theorem 8.A ([24]).** $T$ : *log-hyponormal* $\Longrightarrow T \in$ *class A* $\Longrightarrow T$ : *paranormal.*

On the other hand, Aluthge-Wang introduced the class of $w$-hyponormal operators via Aluthge transformation in [4], and showed an equivalent condition via operator inequalities in [5].

**Definition ([4][5]).**

$$\begin{aligned} T : w\text{-hyponormal} &\Longleftrightarrow |\tilde{T}| \geq |T| \geq |(\tilde{T})^*| \\ &\Longleftrightarrow (|T^*|^{\frac{1}{2}}|T||T^*|^{\frac{1}{2}})^{\frac{1}{2}} \geq |T^*| \text{ and } |T| \geq (|T|^{\frac{1}{2}}|T^*||T|^{\frac{1}{2}})^{\frac{1}{2}}, \end{aligned}$$

where $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transformation of $T = U|T|$.

As a generalization of class A, Fujii-D.Jung-S.H.Lee-M.Y.Lee-Nakamoto [16] introduced class A$(s,t)$ for $s>0$ and $t>0$.

**Definition ([16]).** For $s>0$ and $t>0$,

(i) $T \in$ class A$(s,t) \iff (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$.

(ii) $T \in$ class AI$(s,t) \iff T \in$ class A$(s,t)$ and $T$ is invertible.

**Proposition ([41]).** *Class $A(1,1)$ coincides with class $A$.*

As a generalization of the class of $w$-hyponormal operators, Ito [30] introduced class $w$A$(s,t)$ for $s>0$ and $t>0$ via generalized Aluthge transformation.

**Definition ([30]).** For $s>0$ and $t>0$,

$$\begin{aligned} T \in \text{class } w\mathrm{A}(s,t) &\iff |\tilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t} \text{ and } |T|^{2s} \geq |(\tilde{T}_{s,t})^*|^{\frac{2s}{s+t}} \\ &\iff (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t} \text{ and } |T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}, \end{aligned}$$

where $\tilde{T}_{s,t} = |T|^sU|T|^t$ is the generalized Aluthge transformation of $T = U|T|$.

**Proposition ([30]).** *Class $wA(\frac{1}{2},\frac{1}{2})$ coincides with the class of $w$-hyponormal operators.*

We remark that the following inclusion relations holds for each $s>0$ and $t>0$:

$$\text{class A}(s,t) \supseteq \text{class } w\mathrm{A}(s,t) \supseteq \text{class AI}(s,t).$$

The first relation holds obviously, and the second holds since the first inequality in the definition of class $w$A$(s,t)$ yields the second by the following lemma in case $T$ is invertible.

**Lemma F ([22]).** *Let $A>0$ and $B$ be an invertible operator. Then*

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

*holds for any real number $\lambda$.*

We also remark the following inclusion relations among these classes.

**Theorem 8.B ([16][30]).**

(i) $T$ : *log-hyponormal* $\iff T \in$ *class $AI(s,t)$ for all* $s,t>0$.

(ii) $T$ : *p-hyponormal for some* $p>0$ *or log-hyponormal* $\Longrightarrow T \in$ *class $wA(s,t)$ for all* $s,t>0$.

(iii) $T \in$ *class $A(s,t_1) \Longrightarrow T \in$ class $A(s,t_2)$ for each* $0<s$ *and* $0<t_1\leq t_2$.

(iv) $T \in$ *class $wA(s_1,t_1) \Longrightarrow T \in$ class $wA(s_2,t_2)$ for each* $0<s_1\leq s_2$ *and* $0<t_1\leq t_2$.

As continuation of the study mentioned in §7, Ito [31] and Yamazaki [41] studied powers of invertible class A and class AI$(s,t)$ operators, and Aluthge-Wang [5] and Cho-Huruya-Kim [9] studied powers of $w$-hyponormal operators. As an extension of these results, we obtained the following result on powers of $w\mathrm{A}(s,t)$ operators.

**Theorem 8.1.** *If $T$ belongs to class $wA(s,t)$ for $s \in (0,1]$ and $t \in (0,1]$, then $T^n$ belongs to $wA(\frac{s}{n},\frac{t}{n})$ for all positive integer $n$.*

**Corollary 8.2.** *If $T$ is $w$-hyponormal, then $T^n$ is also $w$-hyponormal for all positive integer $n$.*

We remark that in case $A$ and $B$ are positive and invertible,

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \Longleftrightarrow A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$$

for each $p \geq 0$ and $r \geq 0$ by Lemma F. Very recently, Ito-Yamazaki [33] showed the following result on relations between these two inequalities in case $A$ and $B$ are not necessarily invertible.

**Theorem 8.C ([33]).** *Let $A$ and $B$ be positive operators. Then for each $p \geq 0$ and $r \geq 0$, the following assertions hold:*

(i) *If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, then $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$.*

(ii) *If $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$ and $N(A) \subseteq N(B)$, then $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.*

The following result can be obtained as an immediate corollary of Theorem 8.C.

**Theorem 8.D ([33]).**

(i) *Class $A(s,t)$ coincides with class $wA(s,t)$ for each $s > 0$ and $t > 0$.*

(ii) *Class $A$ coincides with class $wA(1,1)$.*

(iii) *Class $A(\frac{1}{2},\frac{1}{2})$ coincides with the class of $w$-hyponormal operators, i.e., class $wA(\frac{1}{2},\frac{1}{2})$.*

The following result can be obtained by applying Theorem 8.D to (iv) of Theorem 8.B and Theorem 8.1.

**Theorem 8.E ([33]).**

(i) *$T \in$ class $A(s_1,t_1) \Longrightarrow T \in$ class $A(s_2,t_2)$ for each $0 < s_1 \leq s_2$ and $0 < t_1 \leq t_2$.*

(ii) *If $T$ belongs to class $A(s,t)$ for $s \in (0,1]$ and $t \in (0,1]$, then $T^n$ belongs to $A(\frac{s}{n},\frac{t}{n})$ for all positive integer $n$.*

**Corollary 8.F ([33]).** *If $T$ belongs to class $A$, then $T^2$ is $w$-hyponormal.*

# 9 Generalizations of paranormal operators

An operator $T$ is said to be paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in H$. Furuta-Ito-Yamazaki [24] and Fujii-Izumino-Nakamoto [12] introduced absolute-$k$-paranormal and $p$-paranormal operators as generalizations of paranormal operators. We introduced a further generalization called absolute-$(s,t)$-paranormal as parallel concept to class $A(s,t)$.

**Definition ([47]).** For $s > 0$ and $t > 0$,

$$T : \text{absolute-}(s,t)\text{-paranormal} \Longleftrightarrow \| |T|^s |T^*|^t x \|^t \geq \| |T^*|^t x \|^{s+t} \text{ for every unit vector } x \in H$$
$$\Longleftrightarrow \| |T|^s U |T|^t x \|^t \geq \| |T|^t x \|^{s+t} \text{ for every unit vector } x \in H,$$

where the polar decomposition of $T$ is $T = U|T|$.

**Proposition ([47]).** *$T$ is paranormal if and only if $T$ is absolute-$(1,1)$-paranormal.*

A generalization of the result by Ando [6] which states that

$$T : \textit{paranormal} \Longleftrightarrow T^{2*}T^2 - 2\lambda T^*T + \lambda^2 I \geq 0 \textit{ for all } \lambda > 0,$$

we showed a characterization of absolute-$(s,t)$-paranormal operators via operator inequalities.

**Proposition 9.1 ([47]).** *For each $s > 0$ and $t > 0$,*

$$T : \textit{absolute-}(s,t)\textit{-paranormal} \Longleftrightarrow t|T^*|^t |T|^{2s} |T^*|^t - (s+t)\lambda^s |T^*|^{2t} + s\lambda^{s+t} I \geq 0$$
$$\textit{for all } \lambda > 0.$$

We showed the following inclusion relations among these classes. (i) is parallel to (i) of Theorem 8.E, and (ii) is a generalization of the second relation of Theorem 8.A.

**Theorem 9.2 ([47]).**

(i) $T$ : *absolute-$(s_1,t_1)$-paranormal* $\Longrightarrow$ $T$ : *absolute-$(s_2,t_2)$-paranormal for each $0 < s_1 \leq s_2$ and $0 < t_1 \leq t_2$.*

(ii) *For each $s > 0$ and $t > 0$, $T \in$ class $A(s,t)$* $\Longrightarrow$ $T$ : *absolute-$(s,t)$-paranormal.*

(iii) $T$ : *absolute-$(s,t)$-paranormal for some $s, t > 0$* $\Longrightarrow$ $T$ : *normaloid (i.e., $\|T\| = r(T)$).*

We also gave the following characterization of log-hyponormal operators via absolute-$(s,t)$-paranormalities which is an extension of (i) of Theorem 8.B by (ii) of Theorem 9.2.

**Theorem 9.3 ([46][47]).**

$$T : \textit{log-hyponormal} \Longleftrightarrow T : \textit{invertible and } p\textit{-paranormal for all } p > 0$$
$$\Longleftrightarrow T : \textit{invertible and absolute-}(s,t)\textit{-paranormal for all } s > 0 \textit{ and } t > 0.$$

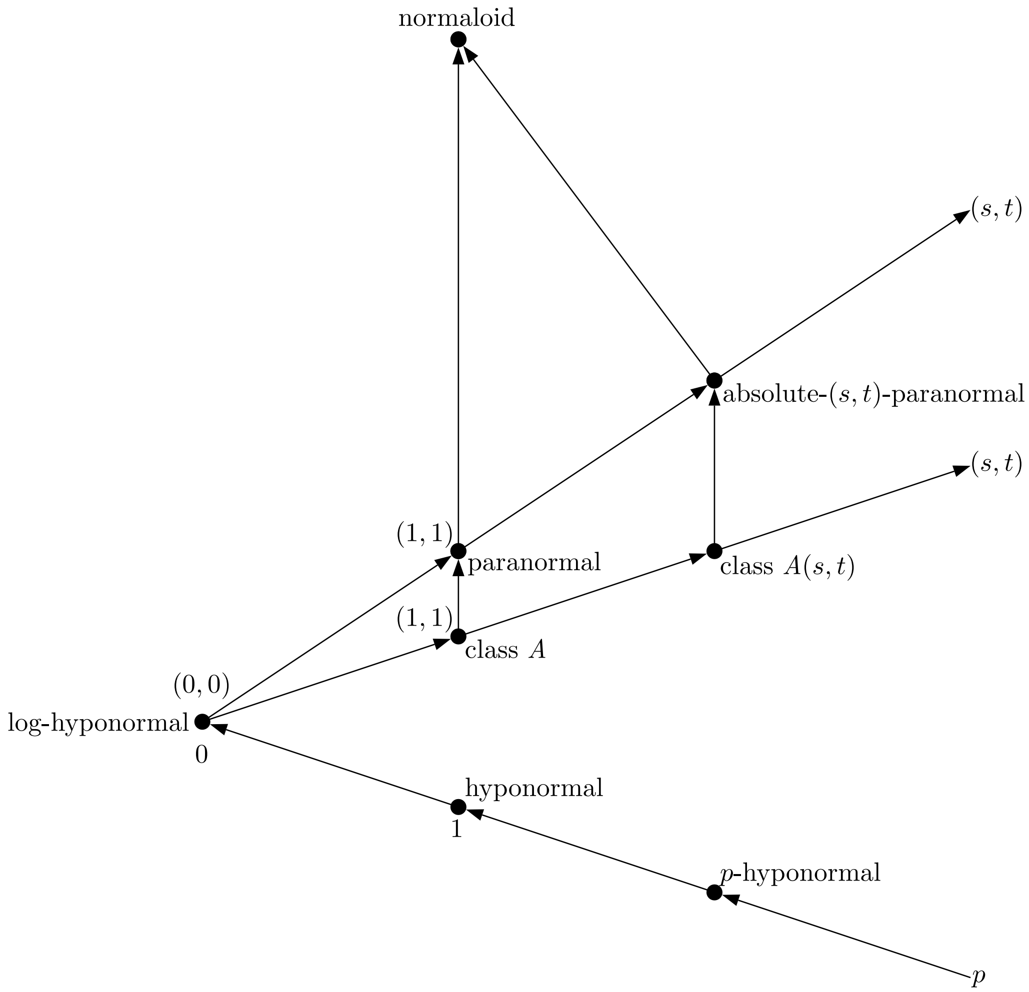
The following result is important to give proofs of the result stated above.

**Theorem 9.A ([36]).** *Let $A$ be a positive operator. Then the following inequalities hold for every unit vector $x \in H$:*

(i) $(A^r x, x) \leq (Ax, x)^r$ *for $0 < r \leq 1$.*

(ii) $(A^r x, x) \geq (Ax, x)^r$ *for $r \geq 1$.*

The following diagram represents the inclusion relations among the classes discussed in this report.




Very recently, Yamazaki and the author [48] showed the following result.

**Theorem 9.4 ([48]).** *Let $A$ and $B$ be positive operators. Then for each $p>0$, $r\geq 0$ and $\lambda>0$, the following assertions hold, where $\frac{X}{Y}$ denotes $XY^{-1}$ for $X\geq 0$ and $Y>0$ in case $XY=YX$:*

(i) *If* $\dfrac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}}+p\lambda^{p+r}I}{(p+r)\lambda^p}\geq B^r$, *then* $A^p\geq\dfrac{(p+r)\lambda^pA^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}}+p\lambda^{p+r}I}$.

(ii) *If* $A^p\geq\dfrac{(p+r)\lambda^pA^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}}+p\lambda^{p+r}I}$ *and* $N(A)\subseteq N(B)$, *then* $\dfrac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}}+p\lambda^{p+r}I}{(p+r)\lambda^p}\geq B^r$.

We remark that in case $A$ and $B$ are positive and invertible,

$$\frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}}+p\lambda^{p+r}I}{(p+r)\lambda^p}\geq B^r\iff A^p\geq\frac{(p+r)\lambda^pA^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}}+p\lambda^{p+r}I}$$

for each $p>0$, $r\geq 0$ and $\lambda>0$.

We also remark that

$$\frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}}+p\lambda^{p+r}I}{(p+r)\lambda^p} \geq (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \quad \text{and} \quad (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}} \geq \frac{(p+r)\lambda^pA^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}}+p\lambda^{p+r}I}$$

hold for each positive invertible operators $A$ and $B$, $p \geq 0$, $r \geq 0$ and $\lambda > 0$ by the arithmetic-geometric-harmonic mean inequality, so that Theorem 9.4 can be understood as a result on the same relations between weaker inequalities than the inequalities in Theorem 8.C.

We showed the following result on normality of absolute-$(s,t)$-paranormal operators as an application of Theorem 9.4.

**Theorem 9.5 ([48]).** *Let $s_1 > 0$, $s_2 > 0$, $t_1 > 0$ and $t_2 > 0$. If $T$ is absolute-$(s_1,t_1)$-paranormal and $T^*$ is absolute-$(s_2,t_2)$-paranormal, then $T$ is normal.*

Theorem 9.5 yields the following corollary on normality of paranormal operators.

**Corollary 9.6 ([48]).** *If $T$ and $T^*$ are paranormal, then $T$ is normal.*

Corollary 9.6 is an extension of the following result by Ando [6].

**Theorem 9.B ([6]).** *If $T$ and $T^*$ are paranormal with $N(T) = N(T^*)$, then $T$ is normal.*

# References


[1] A.Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory **13** (1990), 307–315.

[2] A.Aluthge and D.Wang, *An operator inequality which implies paranormality*, Math. Inequal. Appl. **2** (1999), 113–119.

[3] A.Aluthge and D.Wang, *Powers of p-hyponormal operators*, J. Inequal. Appl. **3** (1999), 279–284.

[4] A.Aluthge and D.Wang, *w-Hyponormal operators*, Integral Equations Operator Theory **36** (2000), 1–10.

[5] A.Aluthge and D.Wang, *w-Hyponormal operators II*, Integral Equations Operator Theory **37** (2000), 324–331.

[6] T.Ando, *Operators with a norm condition*, Acta Sci. Math. (Szeged) **33** (1972), 169–178.

[7] T.Ando, *On some operator inequalities*, Math. Ann. **279** (1987), 157–159.

[8] M.Cho, *On spectra of $AB$ and $BA$*, Proc. KOTAC **3** (2000), 15–19.

[9] M.Cho, T.Huruya and Y.O.Kim, *A note on w-hyponormal operators*, to appear in J. Inequal. Appl.

[10] M.Fujii, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory **23** (1990), 67–72.

[11] M.Fujii, T.Furuta and E.Kamei, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra Appl. **179** (1993), 161–169.

[12] M.Fujii, S.Izumino and R.Nakamoto, *Class of operators determined by the Heinz-Kato-Furuta inequality and the Hölder-McCarthy inequality*, Nihonkai Math. J. **5** (1994), 61–67.

[13] M.Fujii, J.F.Jiang and E.Kamei, *Characterization of chaotic order and its application to Furuta inequality*, Proc. Amer. Math. Soc. **125** (1997), 3655–3658.

[14] M.Fujii, J.F.Jiang and E.Kamei, *A characterization of orders defined by $A^\delta \geq B^\delta$ via Furuta inequality*, Math. Japon. **45** (1997), 519–525.

[15] M.Fujii, J.F.Jiang, E.Kamei and K.Tanahashi, *A characterization of chaotic order and a problem*, J. Inequal. Appl. **2** (1998), 149–156.

[16] M.Fujii, D.Jung, S.H.Lee, M.Y.Lee and R.Nakamoto, *Some classes of operators related to paranormal and log-hyponormal operators*, Math. Japon. **51** (2000), 395–402.

[17] T.Furuta, *On the class of paranormal operators*, Proc. Japan Acad. **43** (1967), 594–598.

[18] T.Furuta, *$A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$*, Proc. Amer. Math. Soc. **101** (1987), 85–88.

[19] T.Furuta, *An elementary proof of an order preserving inequality*, Proc. Japan Acad. Ser. A Math. Sci. **65** (1989), 126.

[20] T.Furuta, *Two operator functions with monotone property*, Proc. Amer. Math. Soc. **111** (1991), 511–516.

[21] T.Furuta, *Applications of order preserving operator inequalities*, Oper. Theory Adv. Appl. **59** (1992), 180–190.

[22] T.Furuta, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl. **219** (1995), 139–155.

[23] T.Furuta, *Invitation to Linear Operators — from matrices to bounded linear operators on a Hilbert space*, Taylor & Francis, Ltd., London, 2001.

[24] T.Furuta, M.Ito and T.Yamazaki, *A subclass of paranormal operators including class of log-hyponormal and several related classes*, Sci. Math. **1** (1998), 389–403.

[25] T.Furuta and M.Yanagida, *On powers of p-hyponormal operators*, Sci. Math. **2** (1999), 279–284.

[26] T.Furuta and M.Yanagida, *On powers of p-hyponormal and log-hyponormal operators*, J. Inequal. Appl. **5** (2000), 367–380.

[27] P.R.Halmos, *A Hilbert Space Problem Book 2nd ed.*, Springer Verlag, New York, 1982.

[28] E.Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415–438.

[29] T.Huruya, *A note on p-hyponormal operators*, Proc. Amer. Math. Soc. **125** (1997), 3617–3624.

[30] M.Ito, *Some classes of operators associated with generalized Aluthge transformation*, SUT J. Math. **35** (1999), 149–165.

[31] M.Ito, *Several properties on class A including p-hyponormal and log-hyponormal operators*, Math. Inequal. Appl. **2** (1999), 569–578.

[32] M.Ito, *Generalizations of the results on powers of p-hyponormal operators*, J. Inequal. Appl. **6** (2001), 1–15.

[33] M.Ito and T.Yamazaki, *Relations between two inequalities $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^p \geq (A^{\frac{p}{2}} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}}$ and their applications*, preprint.

[34] I.B.Jung, E.Ko and C.Pearcy, *Aluthge transforms of operators*, Integral Equations Operator Theory **37** (2000), 437–448.

[35] E.Kamei, *A satellite to Furuta's inequality*, Math. Japon. **33** (1988), 883–886.

[36] C.A.McCarthy, $c_p$, Israel J. Math. **5** (1967), 249–271.

[37] K.Tanahashi, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc. **124** (1996), 141–146.

[38] K.Tanahashi, *On log-hyponormal operators*, Integral Equations Operator Theory **34** (1999), 364–372.

[39] M.Uchiyama, *Some exponential operator inequalities*, Math. Inequal. Appl. **2** (1999), 469–471.

[40] T.Yamazaki, *Extensions of the results on p-hyponormal and log-hyponormal operators by Aluthge and Wang*, SUT J. Math. **35** (1999), 139–148.

[41] T.Yamazaki, *On powers of class $A(k)$ operators including p-hyponormal and log-hyponormal operators*, Math. Inequal. Appl. **3** (2000), 97–104.

[42] T.Yamazaki, *Characterizations of $\log A \geq \log B$ and normaloid operators via Heinz inequality*, to appear in Integral Equations Operator Theory.

[43] T.Yamazaki, *Parallelisms between Aluthge thansformation and powers of operators*, to appear in Acta Sci. Math. (Szeged).

[44] T.Yamazaki, *On numerical range of the Aluthge transformation*, to appear in Linear Algebra Appl.

[45] T.Yamazaki, *An expression of spectral radius via Aluthge transformation*, to appear in Proc. Amer. Math. Soc.

[46] T.Yamazaki and M.Yanagida, *A characterization of log-hyponormal operators via p-paranormality*, Sci. Math. **3** (2000), 19–21.

[47] T.Yamazaki and M.Yanagida, *A further generalization of paranormal operators*, Sci. Math. **3** (2000), 23–31.

[48] T.Yamazaki and M.Yanagida, *Relations between two operator inequalities and their applications to paranormal operators*, preprint.

[49] M.Yanagida, *Some applications of Tanahashi's result on the best possibility of Furuta inequality*, Math. Inequal. Appl. **2** (1999), 297–305.

[50] T.Yoshino, *The p-hyponormality of the Aluthge transform*, Interdiscip. Inform. Sci. **3** (1997), 91–93.