A PARTIAL REGULARITY RESULT FOR HARMONIC MAPS INTO A FINSLER MANIFOLD

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Let N be a n-dimensional C^∞ manifold and TN the tangent bundle of N. We express each point in TN as (u, X) with $u \in N$ and $X \in T_xN$. Moreover, $\mathbf{0}$ denotes the $0-$section $\{(u, 0)\} \subset TN$ when we write $TN \setminus \mathbf{0}$. *A Finsler structure* of N is a function $F : TN \to [0, \infty)$ with the following properties:

(i) Regularity: $F \in C^\infty(TN \setminus \mathbf{0})$.

(ii) Homogeneity: $F(u, \lambda X) = \lambda F(u, X)$ for all $\lambda \geq 0$.

(iii) Convexity: The Hessian matrix of F^2 with respect to X

$$(h_{ij}(u, X)) = \left(\frac{1}{2} \frac{\partial^2 F^2(u, X)}{\partial X^i \partial X^j} \right)$$

is positive definite at every point $(u, X) \in TN \setminus \mathbf{0}$.

We call the pair (N, F) *a Finsler manifold.*

According to Centore [1] the energy density of maps between Finsler manifold is defined as follows.

Definition. *Let (M, G) and (N, F) be Finsler manifolds, and I_xM a unit ball in T_xM, namely*

$$I_xM = \{\xi \in T_xM; G(X) \leq 1\}.$$

For a C^1-map $u : M \to N$, we define the energy density $e(u)(x)$ of u at $x \in M$ by

$$e(u)(x) = \frac{\int_{I_xM} (u^*F)^2(X) dX}{\int_{I_xM} dX}. \tag{0.1}$$

The above definition is consistent with the standard one for Riemannian cases ([1, Lemma 2]) and it can be regarded as a special case of the definition given by Jost [2] for abstract cases of metric measure spaces ([1, Theorem 5]).

In this talk, we consider harmonic maps from $\mathbb{R}^m$ into Finsler space $(N, F) = (\mathbb{R}^n, F)$. For such a case, by virtue of the special structure of $\mathbb{R}^m$ and the homogeneity of F, we can write the energy density defined by (0.1) more simply.

Namely, for a map $u : \mathbb{R}^m \to (N, F)$ we can define the energy density of u at $x \in \mathbb{R}^m$ as

$$e(u)(x) = \frac{1}{2c_m} \int_{S^{m-1}} F^2(u(x), du_x \cdot \xi) d\xi, \tag{0.2}$$

where c_m denotes the area of the $(m-1)$-dimensional standard sphere S^{m-1}, du_x the differential of u at x and

$$du_x \cdot \xi = \left((du_x \cdot \xi)^1, ..., (du_x \cdot \xi)^n\right) = \left(\frac{\partial u^1}{\partial x^\alpha}\xi^\alpha, ..., \frac{\partial u^n}{\partial x^\alpha}\xi^\alpha\right).$$

Moreover, as usual, we define *the energy of u on* $\Omega \subset \mathbb{R}^m$by

$$E(u, \Omega) \quad = \quad \int_\Omega e(u)(x) dx \tag{0.3}$$

We consider minimizing problems for the functional $E(u, \Omega)$ defined by (0.3) in the class

$$H_f^{1,2}(\Omega, N) = \{u \in H^{1,2}(\Omega, N) \; ; \; u - f \in H_0^{1,2}(\Omega, \mathbb{R}^n)\}, \tag{0.4}$$

where $f : \partial\Omega \to N$ is a given L^∞ function. In the following, "a minimizer of E" means "a minimizer of E in the class $H_f^{1,2}(\Omega, N)$".

As in the case that the target manifold is Riemannian, let us call a solution of the Euler-Lagrange equation of (0.3) to be *a harmonic map.*

We consider the following assumption on $h_{ij}(u, X) = \frac{1}{2}\frac{\partial^2 F^2(u,X)}{\partial X^i \partial X^j}$

(h-1) For some concave increasing function ω with $\omega(0) = 0$ we have

$$|h_{ij}(u, X) - h_{ij}(v, X)| \leq \omega(|u - v|) \tag{0.5}$$

for all $u, v, X \in \mathbb{R}^n$.

(h-2) There exists a positive constant λ_0 such that

$$h_{ij}(u, X)\xi^i\xi^j > \lambda_0\|\xi\|^2 \tag{0.6}$$

for all $u, X, \xi \in \mathbb{R}^n$.

Theorem.([3]) *Let $m \leq 4$. Under the above assumptions a minimizer u of $E(u;\Omega)$ is in in the class $C^{1,\alpha}(\Omega_0)$ for some open subset $\Omega_0 \subset \Omega$ with $\mathcal{H}^{m-2-\varepsilon}(\Omega \setminus \Omega_0) = 0$ for some $\varepsilon > 0$.*

References

[1] P.Centore, *Finsler Laplacians and minimal-energy maps,*Int. J. Math. **11** (2000), 1–13.

[2] J. Jost, "Nonpositive curvature: Geometric and analytic aspects", Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, 1997.

[3] A.Tachikawa, *A Partial regularity result for harmonic maps into a Finsler manifold,* to appear in Calc. Var.