EXISTENCE AND REGULARITY FOR THE EVOLUTION OF HIGHER-DIMENSIONAL H−SYSTEMS

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Abstract Let $m \geq 2$ be a positive integer and Ω be a bounded domain in m−dimensional Euclidean space R^m with smooth boundary $\partial\Omega$. For a real number H, the surface of constant mean curvature H in R^{m+1} is prescribed by the nonlinear degenerate elliptic systems of second order partial differential equations, called "H−system",

$$-\mathrm{div}\left(|\nabla u|^{m-2}\nabla u\right) = m^{\frac{m}{2}} H\, \nabla_1 u \wedge \cdots \wedge \nabla_m u \tag{1.1}$$

for a map $u(x) = (u^1(x), \ldots, u^{m+1}(x))$ defined for $x = (x_1, \ldots, x_m) \in \Omega$ with values into R^{m+1}, where $\nabla_\alpha = \frac{\partial}{\partial x^\alpha}$, $\alpha = 1, \cdots, m$, ∇u is the spatial gradient of a map u, $\nabla u = (\nabla_\alpha u^i)$, and the cross product $w_1 \wedge \cdots \wedge w_m : R^{m+1} \oplus \cdots \oplus R^{m+1} \to R^{m+1}$ is defined by the property that $w \cdot w_1 \wedge \cdots \wedge w_m = \det W$ for all vectors $w, w_i \in R^{m+1}$, $i = 1, \ldots, m$, and for the $(m+1) \times (m+1)$−matrix W having the first row $(w^1, \ldots, w^{m+1})$ and the i-th row $(w^1_{i-1}, \ldots, w^{m+1}_{i-1})$, $i = 2, \ldots, m+1$. Here and in what follows, the notation and the summation notation over repeated indices is adopted.

We call a map $u : \Omega \to R^{m+1}$ *conformal* if

$$\nabla_\alpha u \cdot \nabla_\beta u = \lambda^2 \delta_{\alpha\beta} \quad \alpha, \beta = 1, \ldots, m, \tag{1.2}$$

hold in Ω for some real-valued function λ which does not vanish in R^m. If a map u is of C^2−class and conformal, then it is seen that u actually defines a hypersurface $u(\Omega)$ in R^{m+1} which has constant mean curvature H at every point $u(x) \in u(\Omega)$, $x \in \Omega$. Now we consider the Dirichlet boundary value problem for (1.1) with boundary value u_0 which is a given smooth map defined on $\overline{\Omega}$ with values in R^{m+1}.

Note that the equation (1.1) has a variational structure. In fact, a solution of (1.1) gives a surface of least area enclosing a given volume. We call such surfaces as above *soap-bubbles.* We can recognize a solution of the Dirichlet problem for (1.1) to be a critical point of the variational problem, which is to minimize the variational functional, called "m−energy",

$$I(u) = \int_\Omega \tfrac{1}{m} |\nabla u|^m \, dx, \tag{1.3}$$

under the constraint that the quantity, called "volume-functional",

$$V(u) = \tfrac{1}{m+1} \int_\Omega u \cdot \nabla_1 u \wedge \cdots \wedge \nabla_m u \, dx \tag{1.4}$$

is prescribed by $V(u) =$ a given constant, where $V(u)$ is interpreted as the algebraic volume enclosed between the surface $u(\Omega)$ and a fixed surface $u_0(\Omega)$ spanning the curve $u_0(\partial\Omega)$ defined by the Dirichlet data u_0. Observe that (1.1) is the Euler-Lagrange equation of the functional $E(u) = I(u) - m^{\frac{m}{2}} H V(u)$.

The one approach to look for a solution of the Dirichlet boundary value problem for (1.1) is to exploit the evolution for (1.1). Consider the Cauchy-Dirichlet problem: For a map $u : \Omega_\infty = (0, \infty) \times \Omega \longrightarrow R^{m+1}$, $u(z) = (u^1(z), \cdots, u^n(z))$, $z = (t, x) \in \Omega_\infty$,

$$\partial_t u - \mathrm{div}\left(|\nabla u|^{m-2}\nabla u\right) = m^{\frac{m}{2}} H\, \nabla_1 u \wedge \cdots \wedge \nabla_m u \quad \text{in } \Omega_\infty, \tag{1.5}$$

$$u = u_0 \quad \text{on } \{t = 0\} \times \overline{\Omega} \bigcup (0, \infty) \times \partial\Omega. \tag{1.6}$$

$E(u) = I(u) - m^{\frac{m}{2}} HV(u)$. We report a global existence and a regularity of a weak solution of (1.5) and (1.6) for a smooth data having a "small" image. Our main result is the following.

Theorem 1 *Suppose that u_0 be a $W^{1,m}-$map defined on Ω with values in R^{m+1} satisfying the "smallness" condition $|H| \sup_\Omega |u_0| < 1$. Then, there exists a weak solution $u \in L^\infty(0,\infty; W^{1,m}(\Omega, R^{m+1})) \cap W^{1,2}(0,\infty; L^2(\Omega, R^{m+1}))$ of (1.5) and (1.6) such that $\sup_\Omega |u(t)| \leq \sup_\Omega |u_0|$ holds for any $t \geq 0$ and*

$$\int_{(0,T)\times\Omega} |\partial_t u|^2 dz + \sup_{0\leq t\leq T} E(u(t)) \leq E(u_0) \tag{1.7}$$

holds for all $T > 0$. The solution u also satisfies the initial condition, $|u(t) - u_0|_{W^{1,m}(\Omega)} \longrightarrow 0$ as $t \to 0$, and boundary condition $u(t) = u_0$ on $\partial\Omega$ in the trace sense in $W^{1,m}(\Omega, R^{m+1})$ for almost every $t \in (0,\infty)$.

Theorem 2 *Suppose that u_0 be a C^2-map defined on $\overline{\Omega}$ with values in R^{m+1} satisfying the "smallness" condition $|H| \sup_\Omega |u_0| < \frac{1}{2}$. There exist a positive constant $\alpha < 1$ such that the weak solution obtained is locally Hölder continuous in $(0,\infty) \times \overline{\Omega}$ with an exponent α on the parabolic metric $|t|^{\frac{1}{p}} + |x|$ and the Hölder constant is bounded by a positive constant depending only on m, α, $\partial\Omega$, $|H|$ and the C^2-norm of the data u_0. The gradient of the solution is also locally Hölder continuous in $(0,\infty) \times \Omega$ with an exponent α on the usual parabolic metric $|t|^{\frac{1}{2}} + |x|$ and the Hölder constant is bounded by a positive constant depending only on $m, \alpha, |H|$ and $I(u_0)$.*

To prove the existence, we use a time-discrete approximation which consists of the minimization of a family of variational functionals, of which the Euler-Lagrange equations are the time-discrete elliptic partial differential equations of Rothe-type for (1.5). In the study of regularity of a weak solution obtained above, we apply the regularity theorem for the evolutional $p-$Laplacian systems with critical growth on the gradient.

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