Long wave approximations for capillary-gravity waves

Tatsuo IGUCHI

Department of Mathematics, Tokyo Institute of Technology
2-12-1 Oh-okayama, Meguro-ku, Tokyo 152-8551, JAPAN

We are concerned with two-dimensional, irrotational flow of incompressible ideal fluid with free surface under the gravitational field. The domain occupied by the fluid is bounded from below by a solid bottom and above by an atmosphere of constant pressure. The upper surface is free boundary and we take the influence of surface tension into account on the free surface. Our main interest is the motion of the free surface, which is called capillary-gravity wave.

Mathematically, the problem is formulated as a free boundary problem for the incompressible Euler equation with the irrotational condition. After rewriting the equations in an appropriate non-dimensional form, we have two non-dimensional parameters δ and ε the ratio of the water depth h to the wave length l and the ratio of the amplitude of the free surface a to the water depth h, respectively, and another non-dimensional parameter μ called the Bond number, which comes from the surface tension on the free surface. The waves characterized by the physical condition $\delta \ll 1$ are called long waves, but there are several long wave approximations according to relations between ε and δ. For example, we have the following four long wave limits.

(1) The shallow water limit: $\varepsilon = 1$ and $\delta \to 0$.

(2) The Burgers limit: $\varepsilon = \delta \to 0$.

(3) The Korteweg-de Vries limit: $\mu \neq \frac{1}{3}$ and $\varepsilon = \delta^2 \to 0$.

(4) The Kawahara limit: $\mu = \frac{1}{3} + \nu\delta^2$ and $\varepsilon = \delta^4 \to 0$.

In the shallow water limit (1) we obtain the so-called shallow water equations as the limit $\delta \to 0$. The shallow water equations take of the form

$$\begin{cases} u_t + uu_x + \eta_x = 0, \\ \eta_t + \big((1+\eta)u\big)_x = 0. \end{cases}$$

These equations are exactly the same as one-dimensional compressible Euler equation for isentropic flow of a gas of the adiabatic index 2 and its solution generally has a singularity in finite time even if the initial data are sufficiently smooth. In the long wave limit (2), the dynamics of the free surface is approximately translation of two waves without change of the shape, one moving to the right and the other to the left, for a short time interval $0 \le t \le O(1)$. The dynamics of each waves is very slow so that it is invisible for the short time interval. By introducing a slow time scale $\tau = \varepsilon t$, the dynamics can be visible and described by the Burgers equation

$$\pm 2u_\tau + 3uu_x = 0$$

for a long time interval $0 \le t \le O(1/\varepsilon)$. The motion of the free surface in the long wave limit (3) is somewhat similar to that in the limit (2), that is, the dynamics of the free surface is approximately translation of two waves without change of the shape, one moving to the right and the other to the left, for a short time interval $0 \le t \le O(1)$. By introducing a slow time scale $\tau = \varepsilon t$, we see that the dynamics of each waves can be described by the KdV equation

$$\pm 2u_\tau + 3uu_x + \Big(\frac{1}{3} - \mu\Big)u_{xxx} = 0$$

for a long time interval $0 \le t \le O(1/\varepsilon)$. Korteweg and de Vries derived this equation from the equations for capillary-gravity waves in order to prove theoretically the existence of solitary waves. When $\mu = \frac{1}{3}$, this equation degenerates to the Burgers equation. In connection with this critical Bond number, Hasimoto derived a higher-order KdV equation of the form

$$\pm 2u_\tau + 3uu_x - \nu u_{xxx} + \frac{1}{45}u_{xxxxx} = 0$$

in the long wave limit (4), which is nowadays called the Kawahara equation. Historically, this type of equation was first found by Kakutani and Ono in analysis of magnet-acoustic waves in a cold collision free plasma. Then, Hasimoto derived the above equation from capillary-gravity waves. Kawahara studied this type of equation numerically and observed that the equation has both oscillatory and monotone solitary wave solutions.

There are some variants of these long wave limits. If we consider a two-phase flow, then in the long wave limit (2) we obtain the Benjamin-Ono equation

$$\pm 2u_\tau + 3uu_x + \beta H u_{1xx} = 0$$

in place of the Burgers equation, where β is the ratio of the density of upper fluid to the lower one and H is the Hilbert transform defined by

$$Hu(x) = \frac{1}{\pi}\,\mathrm{p.v.}\int_{\mathbf{R}} \frac{u(y)}{x-y}dy.$$

Suppose that there is a background flow and that the bottom is uneven. Let λ be the square of the ratio of the speed of the background flow to the propagation speed of the wave on the free surface, which is called the Froude number. In the critical case $\lambda = 1$ we obtain the forced KdV equation

$$2u_\tau - 3uu_x - \Big(\frac{1}{3} - \mu\Big)u_{xxx} = b'$$

in the long wave limit (3), where the function $b = b(x)$ describes the bottom topography.

In this talk I would like to explain how to give a rigorous justification of the above long wave limits, especially, the forced KdV limit.