Some variational inequality in L^r and its application to the Helmholtz-Weyl decomposition in 3-D bounded domains.

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Let us first impose the following assumption on the domain Ω:

Assumption. Ω is a bounded domain in $\mathbb{R}^3$ with the $C^{2+\mu}$-boundary $\partial\Omega$, where $\mu > 0$.

We denote by $C_{0,\sigma}^\infty(\Omega)$ the set of all C^∞-vector functions $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ with compact support in Ω, such that div $\varphi = 0$. $L^r_\sigma(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ with respect to the L^r-norm $\|\cdot\|_r$; $(\cdot,\cdot)$ denotes the duality pairing between $L^r(\Omega)$ and $L^{r'}(\Omega)$, where $1/r + 1/r' = 1$. $L^r(\Omega)$ stands for the usual (vector-valued) L^r-space over Ω, $1 < r < \infty$. Let us define the space $V^r(\Omega)$ by

$$V^r(\Omega) \equiv \{u \in L^r(\Omega); \text{div } u \in L^r(\Omega), \text{rot } u \in L^r(\Omega), u \times \nu|_{\partial\Omega} = 0\}, \quad 1 < r < \infty. \tag{0.1}$$

It is easy to see that if $u \in L^r(\Omega)$ with rot $u \in L^r(\Omega)$, then it holds $u \times \nu \in W^{1-1/r',r'}(\partial\Omega)^*$. Equipped with the norm $\|u\|_{V^r}$

$$\|u\|_{V^r} \equiv \|\text{div } u\|_r + \|\text{rot } u\|_r + \|u\|_r,$$

we may regard $V^r(\Omega)$ as a closed subset of $W^{1,r}(\Omega)$. Indeed, we have that $V^r(\Omega) \subset W^{1,r}(\Omega)$ with

$$\|\nabla u\|_r \le C\|u\|_{V^r} \quad \text{for all } u \in V^r(\Omega), \tag{0.2}$$

where $C = C(r)$ is a constant depending only on r. Furthermore, we define $V^r_\sigma(\Omega)$ by

$$V^r_\sigma(\Omega) \equiv \{u \in V^r(\Omega); \text{div } u = 0 \quad \text{in } \Omega\}.$$

Finally, we denote by $\mathcal{H}(\Omega)$ the space of harmonic vector fileds on Ω

$$\mathcal{H}(\Omega) \equiv \{h \in C^\infty(\Omega) \cap C^2(\bar{\Omega}); \text{div } h = 0, \text{rot } h = 0 \text{ in } \Omega,\ h \cdot \nu|_{\partial\Omega} = 0\}. \tag{0.3}$$

It is well-known that the dimension of $\mathcal{H}(\Omega)$ is finite. For more precise characterization of $\mathcal{H}(\Omega)$, see Remark 1 (2) below.

Our main result now reads

Theorem 1 *Let Ω be as in the Assumption. Suppose that $1 < r < \infty$. Then for every $u \in L^r(\Omega)$, there are $p \in W^{1,r}(\Omega)$, $w \in V^r_\sigma(\Omega)$ and $h \in \mathcal{H}(\Omega)$ such that u can be represented as*

$$u = h + \text{rot } w + \nabla p. \tag{0.4}$$

Such a triplet $\{p, w, h\}$ is subordinate to the estimate

$$\|\nabla p\|_r + \|w\|_{V^r} + \|h\|_r \le C\|u\|_r \tag{0.5}$$

with the constant $C = C(r)$ independent of u. The above decompostion (0.4) is unique. In fact, if u has another expression

$$u = \tilde{h} + \mathrm{rot}\ \tilde{w} + \nabla\tilde{p}$$

for $\tilde{h} \in \mathcal{H}(\Omega)$, $\tilde{w} \in V^r_\sigma(\Omega)$ and $\tilde{p} \in W^{1,r}(\Omega)$, then we have

$$h = \tilde{h}, \quad \mathrm{rot}\ w = \mathrm{rot}\ \tilde{w}, \quad \nabla p = \nabla\tilde{p}. \tag{0.6}$$

An immediate consequence of the above theorem is

Corollary 1 *Let Ω be as in the Assumption. By the unique decomposition (0.4) we have*

$$L^r(\Omega) = \mathcal{H}(\Omega) \oplus \mathrm{rot}\ V^r_\sigma(\Omega) \oplus \nabla\ W^{1,r}(\Omega), \quad 1 < r < \infty. \quad \textit{(direct sum)} \tag{0.7}$$

Let S_r, R_r and Q_r be projection operators associated to (0.4) from $L^r(\Omega)$ onto $\mathcal{H}(\Omega)$, $\mathrm{rot}\ V^r_\sigma(\Omega)$ and $\nabla\ W^{1,r}(\Omega)$, respectively, i.e.,

$$S_r u \equiv h, \quad R_r u \equiv \mathrm{rot}\ w, \quad Q_r u \equiv \nabla p. \tag{0.8}$$

Then we have

$$\|S_r u\|_r \le C\|u\|_r, \quad \|R_r u\|_r \le C\|u\|_r, \quad \|Q_r u\|_r \le C\|u\|_r \tag{0.9}$$

for all $u \in L^r(\Omega)$, where $C = C(r)$ is the constant depending only on $1 < r < \infty$. Moreover, there holds

$$\left\{\begin{array}{ll} S_r^2 = S_r, & S_r^* = S_{r'}, \\ R_r^2 = R_r, & R_r^* = R_{r'} \\ Q_r^2 = Q_r, & Q_r^* = Q_{r'}, \end{array}\right. \tag{0.10}$$

where S_r^*, R_r^* and Q_r^* denote the adjoint operators on $L^{r'}(\Omega)$ of S_r, R_r and Q_r, respectively.

Remark 1. (1) It is known that

$$L^r(\Omega) = L^r_\sigma(\Omega) \oplus \nabla\ W^{1,r}(\Omega), \quad 1 < r < \infty, \quad (directsum). \tag{0.11}$$

See Fujiwara-Morimoto [4], Solonnikov [11] and Simader-Sohr [9]. Our decomposition (0.7) gives a more precise direct sum of $L^r_\sigma(\Omega)$ such as

$$L^r_\sigma(\Omega) = \mathcal{H}(\Omega) \oplus \mathrm{rot}\ V^r_\sigma(\Omega), \quad 1 < r < \infty. \quad \text{(direct sum)} \tag{0.12}$$

(2) Suppose that the boundary $\partial\Omega$ has $L+1$ connected components $\Gamma_0, \Gamma_1, \cdots, \Gamma_L$ of C^2-surfaces such that $\Gamma_1, \cdots, \Gamma_L$ lie inside of Γ_0 with $\Gamma_i \cap \Gamma_j = \phi$ for $i \ne j$, and scuh that

$$\partial\Omega = \bigcup_{j=0}^{L} \Gamma_j. \tag{0.13}$$

Moreover, we assume that there are N C^2-surfaces $\Sigma_1, \cdots, \Sigma_N$ such that $\Sigma_i \cap \Sigma_j = \phi$ for $i \neq j$, and such that

$$\dot{\Omega} \equiv \Omega \setminus \Sigma, \Sigma \equiv \bigcup_{j=1}^{N} \Sigma_j \quad \text{is simply connected.} \tag{0.14}$$

Then Foias-Temam [3] showed that

$$\dim . \mathcal{H}(\Omega) = N. \tag{0.15}$$

They [3] also gave an orthogonal decompostion of $L^2_\sigma(\Omega)$ such as

$$L^2_\sigma(\Omega) = \mathcal{H}(\Omega) \oplus H_1(\Omega) \quad (\text{orthogonal sum in } L^2(\Omega)),$$

where

$$H_1(\Omega) \equiv \{u \in L^2_\sigma(\Omega); \int_{\Sigma_j} u \cdot \nu dS = 0, \quad j = 1, \cdots, N\}.$$

Yoshida-Giga [13] investigated the operator rot with its domain $D(\text{rot}) = \{u \in H_1(\Omega); \text{rot } u \in H_1(\Omega)\}$ and showed that $H_1(\Omega) = \text{rot } V^2_\sigma(\Omega)$. Futhermore, they [13] gave another type of orthognal L^2-decomposition of vector fileds $u \in D(\text{rot})$. From our decomposition (0.12) with $r = 2$, it follows also that $H_1(\Omega) = \text{rot } V^2_\sigma(\Omega)$.

(3) In the case when Ω is a star-shaped domain, Griesinger [5] gave a simlar decomposition in $L^r(\Omega)$ for $1 < r < \infty$. In her case, it holds $N = 0$. Since she took the smaller space $W^{1,r}_0(\Omega)$ than our space $V^r(\Omega)$, it seems to be an open question whether, in the same way as in (0.7), the anihilator rot $W^{1,r}_0(\Omega)^\perp$ of rot $W^{1,r}_0(\Omega)$ in $L^{r'}(\Omega)$ coinsides with $\nabla\ W^{1,r'}(\Omega)$.

As an application of our decomposition, we have the following gradeint estimates of vector fields via div and rot .

Corollary 2 *Assume that $1 < r < \infty$.*

(1) Let $u \in L^r(\Omega)$ with div $u \in L^r(\Omega)$, rot $u \in L^r(\Omega)$ *and $u \cdot \nu|_{\partial\Omega} = 0$. Then we have $u \in W^{1,r}(\Omega)$ with the estimate*

$$\|\nabla u\|_r \leq C(\|\text{div } u\|_r + \|\text{rot } u\|_r + \|u\|_1), \tag{0.16}$$

where $C = C(r)$ is the constant independent of u.

(2) Let $u \in W^{s,r}(\Omega)$ for $s > 1 + 3/r$ with $u \cdot \nu|_{\partial\Omega} = 0$. Then we have $\nabla u \in L^\infty$ with the estimate

$$\|\nabla u\|_\infty \leq C\left\{1 + \|u\|_r + (\|\text{div } u\|_{bmo} + \|\text{rot } u\|_{bmo}) \log(e + \|u\|_{W^{s,r}})\right\}, \tag{0.17}$$

where $C = C(r)$ is the constant independent of u. For definition of the bmo-norm, see Remark 2 below.

Remark 2. (1) Let us recall the *bmo*-norm in Ω. For $f \in L^1_{loc}(\mathbb{R}^3)$, we define $\|f\|_{bmo(\mathbb{R}^3)}$ by

$$\|f\|_{bmo(\mathbb{R}^3)} = \sup_{x \in \mathbb{R}^3, 0<R<1} \frac{1}{|B_R(x)|} \int_{B_R(x)} |f(y) - f_{B_R(x)}|dy + \sup_{x \in \mathbb{R}^3} \frac{1}{|B_1(x)|} \int_{B_1(x)} |f(y)|dy$$

with $f_{B_R(x)} = \dfrac{1}{|B_R(x)|}\displaystyle\int_{B_R(x)} f(y)dy$, where $B_R(x)$ denotes the ball in $\mathbb{R}^3$ centered at x with radius R and $|B_R(x)|$ is its volume. For $g \in L^1_{loc}(\Omega)$ we say $g \in bmo(\Omega)$ if there is an extension $f \in bmo(\mathbb{R}^3)$ such that $g = f$ on Ω. The bmo-norm $\|g\|_{bmo}$ of g on Ω is defined by

$$\|g\|_{bmo} \equiv \inf\{|f\|_{bmo(\mathbb{R}^3)}; f \in bmo(\mathbb{R}^3), f = g \quad \text{on } \Omega\}.$$

(2) von Wahl [12] proved that (0.16) without $\|u\|_1$ on the right hand side holds if and only if $N = 0$, i.e., Ω is simply connected. He also showed the same estimate for $u \in W^{1,r}(\Omega)$ with $u \times \nu = 0$ on $\partial\Omega$ if and only if $L = 0$. Our variational inequality makes it possible to prove (0.16) also for $u \in W^{1,r}(\Omega)$ with $u \times \nu = 0$ on $\partial\Omega$. von Wahl's estimate [12] may be regarded as a special case of ours sicnce we can treat the general case such as (0.13) and (0.14). His method is based on the representation formula for $u \in W^{1,r}(\Omega)$ via div u and rot u which is different from ours.

(3) In $\mathbb{R}^3$, by means of the Biot-Savard law, Beale-Kato-Majda [1] and Kozono-Taniuchi [6] obtained a similar estimate to (0.17) for $u \in W^{s,r}(\mathbb{R}^3)$ with $s > 1 + 3/r$. More generalized version in the homogeneou Besov space $\dot{B}^0_{\infty,\infty}$ is found in Kozono-Ogawa-Taniuch [7]. In the case of simply connected bounded domains Ω in $\mathbb{R}^3$, Ferrari showed (0.17) for div $u = 0$ with $u \cdot \nu|_{\partial\Omega} = 0$. More general case such as (0.13) and (0.14) was treated by Shirota-Yanagisawa [10] and Ogawa-Taniuchi [8].

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