Global Attractor for the Complex Ginzburg-Landau Equation

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Let Ω be a "bounded" domain in $\mathbb{R}^N (N \in \mathbb{N})$ with C^2-boundary $\partial\Omega$. We consider the existence of a global attractor for the C_0 semigroup associated with the initial-boundary value problem for the complex Ginzburg-Landau equation:

$$\text{(CGL)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 & \text{in } \Omega \times \mathbb{R}_+, \\ u(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & \text{on } \partial\Omega, \end{cases}$$

where $\lambda, \kappa \in \mathbb{R}_+ := (0, \infty)$, $\alpha, \beta, \gamma \in \mathbb{R}$, and $q \geq 2$ are constants, and u is a complex-valued unknown function. We consider (CGL) in $X := L^2(\Omega)$, assuming that

$$2 \leq q \leq 2 + 4/N.$$

Then for all $u_0 \in X$ there exists a unique strong solution $u = u(t)$ to (CGL) such that

$$||u(t) - v(t)||_X \ \leq \ e^{K_1 t + K_2 e^{2\gamma_+ t}(||u_0||_X \vee ||v_0||_X)^2} ||u_0 - v_0||_X \quad \forall\, t \geq 0,$$

where $\gamma_+ := \max\{0, \gamma\}$ and K_1, K_2 are constants. Setting $U(t)u_0 := u(t)$, the family $\{U(t); t \geq 0\}$ forms a C_0 *semigroup* on X (see [**1**]).

To state our result we need the notion of the definition of a global attractor.

Definition. $\mathcal{A} \subset X$ is a *global attractor* for $\{U(t)\}$ if and only if

(a) $\mathcal{A}$ is a non-empty compact set;

(b) $\mathcal{A}$ is invariant under $\{U(t)\}$: $U(t)\mathcal{A} = \mathcal{A}$, $\forall\, t \geq 0$;

(c) $d(U(t)B, \mathcal{A}) := \sup\limits_{x \in U(t)B} \inf\limits_{y \in \mathcal{A}} ||x - y||_X \to 0$ as $t \to +\infty$, $\forall\, B \subset X$ (bounded).

Our result is stated as follows:

Theorem. *Let $\{U(t)\}$ be a C_0 semigroup on $L^2(\Omega)$ as stated above. Assume that either $2 < q \leq 2 + 4/N$, or $q = 2$ with $\gamma < \kappa$. Then $\{U(t)\}$ has a global attractor in $L^2(\Omega)$.*

To prove the theorem we use the abstract formulation in a complex Hilbert space X:

$$\text{(ACP)} \qquad \begin{cases} \dfrac{du}{dt} + (\lambda + i\alpha)Su + (\kappa + i\beta)\partial\psi(u) - \gamma u = 0, \\ u(0) = u_0, \end{cases}$$

where S is a nonnegative self-adjoint operator in X and $\partial\psi$ is the subdifferential of a lower semi-continuous convex function ψ on X, with the same constants $\lambda, \kappa \in \mathbb{R}_+$ and $\alpha, \beta \in \mathbb{R}$ as in (CGL). Under certain conditions, we can show that the C_0 semigroup and the global attractor of (ACP) also exist.

References

[1] N. Okazawa and T. Yokota, Non-contraction semigroups generated by the complex Ginzburg-Landau equation, GAKUTO International Series, Math. Sci. Appl., vol. 20, Gakkōtosho, Tokyo, 2004, pp. 490–504.