Asymptotic Estimates for the Spectral Gaps of the Schrödinger Operators with Periodic δ'-Interactions

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In this talk we discuss the spectrum of the Schrödinger operator which is formally expressed as

$$H = -\frac{d^2}{dx^2} + \sum_{l=-\infty}^{\infty} \big(\beta_1 \delta'(x - 2\pi l) + \beta_2 \delta'(x - \kappa - 2\pi l)\big) \quad \text{in} \quad L^2(\mathbf{R}),$$

where $\kappa \in (0, 2\pi)$ and $\beta_1,\ \beta_2 \in \mathbf{R} \setminus \{0\}$ are parameters, the symbol $'$ stands for the derivative with respect to x, and $\delta(x)$ is the Dirac δ-function at the origin. The precise definition of this operator is given through boundary conditions as follows. Let

$$Z_1 = 2\pi\mathbf{Z}, \qquad Z_2 = \{\kappa\} + 2\pi\mathbf{Z}, \qquad Z = Z_1 \cup Z_2,$$

and

$$A_l = \begin{pmatrix} 1 & \beta_l \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad l = 1, 2.$$

We define

$$(Hy)(x) = -y''(x), \qquad x \in \mathbf{R} \setminus Z,$$

$$\mathrm{Dom}(H) = \Big\{ y \in H^2(\mathbf{R} \setminus Z)\ ; \\ \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = A_l \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \quad \text{for} \quad x \in Z_l, \quad l = 1, 2 \Big\}.$$

In order to formulate our main result, we recall basic spectral properties of H from [10]. The operator H is self-adjoint. Let us consider the equations

$$\begin{cases} -y''(x) = \lambda y(x), \qquad x \in \mathbf{R} \setminus Z, \\ \begin{pmatrix} y(x+0) \\ y'(x+0) \end{pmatrix} = \begin{pmatrix} 1 & \beta_l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(x-0) \\ y'(x-0) \end{pmatrix} \quad \text{for} \quad x \in Z_l, \quad l = 1, 2, \end{cases} \tag{1}$$

where λ is a complex parameter. By $y_1(x, \lambda)$ and $y_2(x, \lambda)$ we denote the solutions of (1) subject to the initial conditions

$$(y_1(+0, \lambda),\ y_1'(+0, \lambda)) = (1,\ 0)$$

and

$$(y_2(+0, \lambda),\ y_2'(+0, \lambda)) = (0,\ 1),$$

respectively. We introduce the discriminant of the equations (1):

$$D(\lambda) = y_1(2\pi + 0, \lambda) + y_2'(2\pi + 0, \lambda),$$

which is an entire function. All the zeros of $D(\cdot) \mp 2$ are real, and they form an increasing sequence which diverges to $+\infty$. For $j \in \mathbf{N} = \{1, 2, 3, \cdots\}$, we denote by $\lambda_j^{\pm}$ the j-th zero of $D(\cdot) \mp 2$ counted with multiplicity. Then we have

$$\lambda_1^{\mp} < \lambda_1^{\pm} \leq \lambda_2^{\pm} < \lambda_2^{\mp} \leq \lambda_3^{\mp} < \cdots < \lambda_{2k-1}^{\pm} \leq \lambda_{2k}^{\pm} < \lambda_{2k}^{\mp} \leq \lambda_{2k+1}^{\mp} < \cdots$$

for $\pm\beta_1\beta_2 < 0$ (see Proposition 1(d), (e) of [10]). For $\pm\beta_1\beta_2 < 0$, we define

$$B_j = \begin{cases} [\lambda_j^{\mp},\ \lambda_j^{\pm}] & \text{if } j \text{ is odd,} \\ [\lambda_j^{\pm},\ \lambda_j^{\mp}] & \text{if } j \text{ is even,} \end{cases}$$

$$G_j = \begin{cases} (\lambda_j^{\pm},\ \lambda_{j+1}^{\pm}) & \text{if } j \text{ is odd,} \\ (\lambda_j^{\mp},\ \lambda_{j+1}^{\mp}) & \text{if } j \text{ is even.} \end{cases}$$

The spectrum of H is then given by

$$\sigma(H) = \bigcup_{j=1}^{\infty} B_j.$$

The closed interval B_j is called the j-th band of $\sigma(H)$, the open interval G_j the j-th gap.

The aim of this talk is to analyze the asymptotic behavior of $|G_j|$, the length of the j-th gap of the spectrum of H, as $j \to \infty$. We impose the following assumption on κ.

$$(\mathbf{A.1}) \qquad \frac{\kappa}{2\pi} = \frac{m}{n}, \quad (m, n) \in \mathbf{N}^2 \quad \text{and} \quad \gcd(m, n) = 1.$$

We further assume that the prime period of the interactions is 2π, i.e.,

$$(\mathbf{A.2}) \qquad \text{either} \quad (m, n) \neq (1, 2) \quad \text{or} \quad \beta_1 \neq \beta_2 \quad \text{holds.}$$

Let

$$a_k = \frac{n}{2m} k \qquad \text{for} \qquad k = 1, 2, \ldots, m-1,$$
$$b_l = \frac{n}{2(n-m)} l \qquad \text{for} \qquad l = 1, 2, \ldots, n-m-1.$$

Let

$$c_1 < c_2 < \cdots < c_{n-2}$$

be the rearrangement of the elements of $\{a_k\}_{k=1}^{m-1} \cup \{b_l\}_{l=1}^{n-m-1}$. We set $c_0 = 0$, $c_{n-1} = n/2$, and

$$d_k = c_k - c_{k-1} \qquad \text{for} \qquad k = 1, 2, \ldots, n-1.$$

Our main result is stated as follows.

THEOREM 1. *Adopt the assumptions* (**A.1**) *and* (**A.2**).
(i) *For each* $k \in \{1, 2, \ldots, n-1\}$, *we have*

$$|G_{nj+1+k}| = n d_k j + O(1) \qquad \text{as} \quad j \to \infty.$$

(ii) *If* $\beta_1 \beta_2 < 0$, *then*

$$|G_{nj+1}| = \left| \frac{4(\beta_1 + \beta_2)\pi}{\beta_1 \beta_2 \kappa (2\pi - \kappa)} \right| + O(j^{-1}) \qquad \text{as} \quad j \to \infty.$$

(iii) *If* $\beta_1 \beta_2 > 0$, *then*

$$|G_{nj+1}| = \frac{4\sqrt{(\beta_1 + \beta_2)^2 \pi^2 - 4\beta_1 \beta_2 \kappa (2\pi - \kappa)}}{\beta_1 \beta_2 \kappa (2\pi - \kappa)} + O(j^{-1}) \qquad \text{as} \quad j \to \infty.$$

References

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*, Springer, Heidelberg, 1988.

[2] S. Albeverio and P. Kurasov, *Singular perturbations of differential operators*, London Mathematical Society Lecture Note Series 271, Cambridge Univ. Press, 1999.

[3] F. Gesztesy and H. Holden, A new class of solvable models in quantum mechanics describing point interactions on the line, J. Phys. A: Math. Gen. **20** (1987), 5157-5177.

[4] F. Gesztesy, H. Holden and W. Kirsch, On energy gaps in a new type of analytically solvable model in quantum mechanics, J. Math. Anal. Appl. **134** (1988), 9-29.

[5] G. Sh. Guseinov and I. Y. Karaca, Instability intervals of a Hill's equation with piecewise constant and alternating coefficient, Comput. Math. Appl. **47** (2004), 319-326.

[6] R. Hughes, Generalized Kronig-Penney Hamiltonians, J. Math. Anal. Appl. **222** (1998), 151-166.

[7] T. Kappeler and C. Möhr, Estimates for periodic and Dirichlet eigenvalues of the Schrödinger operator with singular potentials, J. Funct. Anal. **186** (2001), 62-91.

[8] R. Kronig and W. Penney, Quantum mechanics in crystal lattices, Proc. Royal Soc. London **130** (1931), 499-513.

[9] W. Magnus and S. Winkler, *Hill's equations*, Wiley, New York, 1966.

[10] K. Yoshitomi, Spectral gaps of the one-dimensional Schrödinger operators with periodic point interactions, Hokkaido Math. J., to appear.