Perturbation theorem for C_0-groups on Hilbert space

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We consider perturbations of C_0-group generators. Let X be a complex Hilbert space. First, we define a C_0-group on X and its generator.

Definition 1. (**i**) A one-parameter family $\{T(t); t \in \mathbb{R}\}$ is a *C_0-group* of bounded linear operators if (a) $T(0) = I$; (b) $T(t+s) = T(t)T(s) \ \forall\, t, s \in \mathbb{R}$; (c) $T(t)x \to x \ (t \to 0) \ \forall\, x \in X$.
(**ii**) *The generator* A of a C_0-group $\{T(t); t \in \mathbb{R}\}$ is defined as

$$y = Ax \ (x \in D(A)) \iff T(t)x - x = \int_0^t T(s)y ds.$$

An operator A is the generator of a C_0-group $\{T(t); t \in \mathbb{R}\}$ satisfying $\|T(t)\| \leq Me^{\gamma|t|} \ \forall\, t \in \mathbb{R}$ if and only if A is closed, $\overline{D(A)} = X$ and $\|(\lambda \pm A)^{-n}\| \leq M(\lambda - \gamma)^{-n} \ \forall\, \mathrm{Re}\,\lambda > \gamma \geq 0$ and $n \in \mathbb{N}$.

The next theorem is concerned with perturbations of the generator of a C_0-group, however, it involves only first-order resolvent condition.

Theorem 2. *Let A be the generator of a C_0-group on X and B a closed operator on X such that $D(A) \subset D(B)$. Assume that there exist constants $0 < M < 1$ and $\gamma_0 > 0$ such that*

$$\|B(\gamma + i\,\omega - A)^{-1}\| \leq M, \tag{1}$$

$$\|(\gamma + i\,\omega - A)^{-1}Bu\| \leq M\|u\| \quad \forall\, u \in D(B), \tag{2}$$

$$\|B(\gamma + i\,\omega + A)^{-1}\| \leq M, \tag{3}$$

for all $\gamma > \gamma_0$ and $\omega \in \mathbb{R}$ (here $i = \sqrt{-1}$). Then $A + B$ generates a C_0-group on X.

Referring to [**1**] and [**2**], we can prove this theorem.
1st step.(1) and (2) imply by [**1**] that $A + B$ generates a C_0-semigroup $\{T(t); t \geq 0\}$ on X.
2nd step.(3) guarantees by [**2**] that $\{T(t)\}$ is embedded into a C_0-group on X.

The conclusion is thus trivial if we assume further that

$$\|(\gamma + i\,\omega + A)^{-1}Bu\| \leq M\|u\| \quad \forall\, u \in D(B). \tag{4}$$

Therefore the meaning of Theorem 1 may be stated as follows: only condition (3) is sufficient to extend the semigroup to a group without having recourse to (4).

This theorem is applied to Schrödinger type operators as follows.
Example. Let $X := L^2(\mathbb{R})$, $k \in \mathbb{N}$ and consider differential operators of the form:

$$(Au)(x) := i\,\frac{d^{2k}u}{dx^{2k}}(x), \quad D(A) := H^{2k}(\mathbb{R}),$$

$$(Bv)(x) := V(x)\frac{d^l v}{dx^l}(x), \quad D(B) := H^l(\mathbb{R}),$$

where $V \in H^{l+1}(\mathbb{R})$ and $l = 0, 1, 2, \ldots, k-1$. Then $A + B$ generates a C_0-group on $L^2(\mathbb{R})$.

References

[1] C.J.K. Batty, On a perturbation theorem of Kaiser and Weis, Semigroup Forum **70** (2005), 471–474.

[2] K. Liu, A characterization of strongly continuous groups of linear orepators on a Hilbert space, Bulletin of London Math. Soc. **32** (2000), 54–62.