L^p-theory of second-order elliptic operators with unbounded coefficients

Hiroshi Tamura
(Tokyo University of Science)

Let $p \in (1, \infty)$ and $N \in \mathbf{N}$. In the complex Banach space L^p $(:= L^p(\mathbf{R}^N))$, we consider the elliptic partial differential operator of the form

$$Tu := -\Delta u + b \cdot \nabla u + qu.$$

Here $b := (b_1, b_2, \ldots, b_N)$ is a real-vector-valued function on $\mathbf{R}^N$ and q is a complex-valued function on $\mathbf{R}^N$. We assume that

$$\text{(A1)} \qquad (1+|x|)^{-1} b_j \in L^{2p} \cap L^\infty, \ \frac{\partial b_j}{\partial x_k} \in L^\infty \ \ (j, k = 1, 2, \ldots, N),$$

$$\text{(A2)} \qquad (1+|x|)^{-1} q \in L^\infty, \ \mathrm{Re}\ q \geq 0, \ \frac{\partial q}{\partial x_k} \in L^\infty \ \ (k = 1, 2, \ldots, N).$$

Let $T_{\min}$ be the restriction of the formal differential expression T on $C_0^\infty(\mathbf{R}^N)$.
Under these conditions we obtain the following

Theorem. *$T_{\min}$ is closable and its closure $\widetilde{T}_{\min}$ is quasi-m-accretive in L^p. More precisely, for some $\lambda \in \mathbf{R}$,*

$$\mathrm{Re}((T_{\min} + \lambda)u, |u|^{p-2}u) \geq 0, \ \ u \in C_0^\infty(\mathbf{R}^N),$$
$$R(\widetilde{T}_{\min} + \lambda) = L^p.$$

Kato [1] presented this problem in L^2, and Noguchi [2] replaced Δu with $\mathrm{div}(a \nabla u)$. Conditions (A1) and (A2) are stronger than those in [1]. To prove this theorem we introduce the intermediate operator T_{int} whose definition is different from that in [1].

Definition. Define T_{int} as

$$\begin{cases} D(T_{\mathrm{int}}) := \{u \in W^{1,1}(\mathbf{R}^N) \cap W^{1,\infty}(\mathbf{R}^N) \cap W^{2,p}_{\mathrm{loc}}(\mathbf{R}^N); \\ \qquad\qquad (1+|x|)^{-1} B_j u \in L^p \ (j = 1, 2, \ldots, N), \ Tu \in L^p\}, \\ T_{\mathrm{int}} u := Tu, \ \ u \in D(T_{\mathrm{int}}). \end{cases}$$

Here B_j is formally defined as

$$B_j u := -2 \frac{\partial u}{\partial x_j} + b_j u, \qquad (j = 1, 2, \ldots, N).$$

Then we present the key propositions in terms of this operator.

Proposition 1. $T_{\min} \subset T_{\mathrm{int}} \subset \widetilde{T}_{\min}$.

Proposition 2. $C_0^\infty(\mathbf{R}^N) \subset R(T_{\mathrm{int}} + \lambda)$ *for some* $\lambda \in \mathbf{R}$.

To prove these propositions we need cut-off functions presented by Kato [1]. By virtue of the functions we can give the growth rate of $(1+|x|)$-order to b and q.

References

[1] T. Kato, Nonselfadjoint Schrödinger operators with singular first-order coefficients, Proc. Roy. Soc. Edinburgh, Sect. A **96** (1984), 323-329.
[2] 野口大介, L^2-theory of second-order elliptic operators with singular first-order coefficients, 修士論文, 2007.