Abstract approach to Schrödinger evolution equations

Kentarou Yoshii
(Tokyo University of Science)

Let $\{A(t); 0 \le t \le T\}$ be a family of closed linear operators in a complex Hilbert space X. We are concerned with linear evolution equations of the form

$$\text{(E)} \qquad \frac{d}{dt}u(t) + A(t)u(t) = f(t) \quad \text{on} \quad (0,T).$$

Let S be a self-adjoint operator in X, satisfying $(u, Su) \ge \|u\|^2$ for $u \in D(S)$. Then the square root $S^{1/2}$ is well-defined. Put $Y := D(S^{1/2})$ and $(u,v)_Y := (S^{1/2}u, S^{1/2}v),\ u, v \in Y$. Then Y is a Hilbert space with norm $\|v\|_Y := (v,v)_Y^{1/2}$ embedded continuously and densely in X. For $\{A(t)\}$ and S assume that

(I) There is $\alpha \in L^1(0,T)$, $\alpha \ge 0$, such that

$$|\operatorname{Re}(A(t)v, v)| \le \alpha(t)\,\|v\|^2,\ v \in D(A(t)),\ \text{a.a. } t \in (0,T).$$

(II) $Y \subset D(A(t)),\ \text{a.a. } t \in (0,T)$.
(III) There is $\beta \in L^1(0,T)$, $\beta \ge \alpha$, such that

$$|\operatorname{Re}(A(t)u, Su)| \le \beta(t)\,\left\|S^{1/2}u\right\|^2,\ u \in D(S),\ \text{a.a. } t \in (0,T).$$

(IV) $A(\cdot) \in L^2\big(0,T; B(Y,X)\big)$.

Under the assumption stated above we can prove the following

Theorem. *Let $f(\cdot) \in L^2(0,T;X) \cap L^1(0,T;Y)$. Then there exists a unique strong solution $u(\cdot)$ of* (E) with $u(0) = u_0 \in Y$ such that $u(\cdot) \in H^1(0,T;X) \cap C([0,T];Y)$.

In particular, if $A(\cdot)$ is strongly continuous on $[0,T]$ to $B(Y,X)$ and α, β are constants, then Theorem has already been proved in Okazawa [1].

Now let $n \in \mathbb{N}$. By introducing $S := 1 + \Delta^{2n} + |x|^{4n}$ we can apply the above-mentioned theorem to the Cauchy problem for Schrödinger evolution equations:

$$\text{(SE)} \qquad i\frac{\partial u}{\partial t}(x,t) - \big(-\Delta_x + V(x,t)\big)u(x,t) = 0, \quad \text{a.a. t} \in (0,\infty)$$

in $L^2(\mathbb{R}^N)$. The assumption is satisfied under the following conditions:
(V0) $V(\cdot,t) \in C^{2n}(\mathbb{R}^N)$ a.a. t $\in (0,\infty)$.
(V1) There is $g_0 \in L^2_{\text{loc}}[0,\infty)$ such that $|V(x,t)| \le g_0(t)(1+|x|^{2n})$.
(V2) There are $g_j \in L^1_{\text{loc}}[0,\infty)$ $(1 \le j \le 2n)$ such that

$$\begin{cases} \displaystyle\sum_{|\alpha|=j} \left|D_x^\alpha V(x,t)\right| \le g_j(t)(1+|x|^j), & (1 \le j \le n), \\ \displaystyle\sum_{|\alpha|=n} \left|D_x^\alpha \Delta_x^{\frac{j-n}{2}} V(x,t)\right| \le g_j(t)(1+|x|^j), & (n < j \le 2n,\ j-n : \text{even}), \\ \displaystyle\sum_{|\alpha|=n-1} \left|D_x^\alpha \Delta_x^{\frac{j-n+1}{2}} V(x,t)\right| \le g_j(t)(1+|x|^j), & (n < j \le 2n,\ j-n : \text{odd}). \end{cases}$$

References

[1] N. Okazawa, Remarks on linear evolution equations of hyperbolic type in Hilbert space, Adv. Math. Sci. Appl. **8** (1998), 399–423.