Approximation theorem for evolution equations of hyperbolic type in Hilbert space

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Let X be a (complex) Hilbert space. Let $\{A(t); 0 \le t \le T\}$ be a family of closed linear operators in X. We are concerned with the linear evolution equation

$$(\mathbf{E}) \qquad \begin{cases} \dfrac{du(t)}{dt} + A(t)u(t) = f(t) \quad \text{on} \quad [0,T], \\ u(0) = u_0. \end{cases}$$

Let S be a selfadjoint operator with domain $D(S)$ in X, satisfying

$$(u, Su) \ge \|u\|^2 \quad \text{for} \quad u \in D(S).$$

Then the square root $S^{\frac{1}{2}}$ is well-defined and its domain $Y := D(S^{\frac{1}{2}})$ is regarded as a Hilbert space with inner product $(S^{\frac{1}{2}}u, S^{\frac{1}{2}}v)$. In what follows we denote by $B(Y,X)$ the set of all bounded linear operators on Y to X. Assume that $A(t)$ satisfies the conditions (I), (II) and (III) stated in [**1**] or [**2**]. The solvability of (**E**) is proved in [**2**] under the condition that $A(\cdot) \in C([0,T]; B(Y,X))$.

In this paper, we consider the **approximate problem** to (**E**):

$$(\mathbf{E}_\varepsilon) \qquad \begin{cases} \dfrac{du_\varepsilon(t)}{dt} + A_\varepsilon(t)u_\varepsilon(t) = f(t) \quad \text{on} \quad (0,T], \\ u_\varepsilon(0) = u_0, \end{cases}$$

where $A_\varepsilon(t) := A(t) + \varepsilon S$. Under conditions (I)–(III), $-A_\varepsilon(t)$ is a generator of an analytic semigroup on X. In this sense, ($\mathbf{E}_\varepsilon$) is an equation of parabolic type, while (**E**) is of hyperbolic type. Namely, ($\mathbf{E}_\varepsilon$) is a parabolic regularization of the hyperbolic problem (**E**). We have tried to make it clear that each solution to (**E**) is a limit of the corresponding family of solutions to ($\mathbf{E}_\varepsilon$). Thus we have the following

Theorem. *Assume that there are two constants* $1/2 < \alpha \le 1$, $0 < \beta \le 1$ *such that* $A(\cdot) \in C^{0,\alpha}([0,T]; B(Y,X))$ *and* $f(\cdot) \in C^{0,\beta}([0,T]; X) \cap L^1(0,T;Y)$. *Let* $u(t)$ *be a solution to* (**E**). *Then the solutions* $u_\varepsilon(t)$ *to* ($\mathbf{E}_\varepsilon$) *exist and converge to* $u(t)$, *i.e.,*

$$u_\varepsilon(t) \to u(t) \quad (\varepsilon \downarrow 0) \quad \text{in} \quad C([0,T]; X).$$

Note that $C^{0,\alpha}([0,T]; B(Y,X))$ is the space of all Hölder continuous $B(Y,X)$-valued functions on $[0,T]$. In the Lipschitz case (in which $\alpha = \beta = 1$) the convergence of $u_\varepsilon(t)$ to $u(t)$ is already studied in [**1**].

References

[1] N. Okazawa and A. Unai, Singular perturbation approach to evolution equations of hyperbolic type in Hilbert space, Adv. Math. Sci. Appl. **3** (1993/94), 267–283.

[2] N. Okazawa and A. Unai, Linear evolution equations of hyperbolic type in Hilbert space, SUT J. Math. **29** (1993), 51–70.