Drift-diffusion System in the Critical Cases

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1 Well posedness of the critical drift-diffusion system in two dimensions

We [1] consider the two dimensional drift-diffusion system in $\mathbb{R}^2$:

$$\begin{cases} \partial_t n - \Delta n + \nabla \cdot (n\nabla\psi) = 0, & t>0, x\in\mathbb{R}^2, \\ \partial_t p - \Delta p - \nabla \cdot (p\nabla\psi) = 0, & t>0, x\in\mathbb{R}^2, \\ -\Delta\psi = \kappa(p-n), \quad x\in\mathbb{R}^2, \\ \quad n(0,x)=n_0(x), \qquad p(0,x)=p_0(x). \end{cases} \tag{1.1}$$

The existence and well-posedness of the solution of the drift-diffusion system (1.1) is obtained by the integral formula via the semigroup representation with a contraction mapping argument. See for example, Kurokiba-Ogawa [25]. This method is also valid for the two dimensional simplified Keller-Segel system:

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u\nabla\psi) = 0, & t>0, x\in\mathbb{R}^2, \\ -\Delta\psi = \kappa u, \quad x\in\mathbb{R}^2, \\ \quad u(0,x)=u_0(x). \end{cases} \tag{1.2}$$

The basic function space for x variable is $L^p(\mathbb{R}^n)$, where $\frac{n}{2}<p<n$, $n=2,3$ and this framework is analogous to the result for the Navier-Stokes system. While the energy method works also well and we may derive the local well-posedness for two dimensional case critical case $p=2$ (see for this case [24]). Note that we need to introduce a weighted $L^2(\mathbb{R}^n)$ class since we need to controle the solution of the Poisson equation in two dimensions.

On the other hand, if we consider the other critical case $p=1$ by the method of the integral equation, we should emphasize that the system (1.1) or simpler model (1.2) has the similar scaling structure of the system as the two dimensional vortex equation of the Navier-Stokes equation:

$$\begin{cases} \partial_t \omega - \Delta\omega + u\nabla\omega = 0, & t>0, x\in\mathbb{R}^2, \\ -\Delta u = \mathrm{rot}\ \omega = \partial_1\omega_2 - \partial_2\omega_1, \quad x\in\mathbb{R}^2, \\ \quad \omega(0,x) = \mathrm{rot}\ u_0(x). \end{cases} \tag{1.3}$$

When we consider the two dimensional vortex equation (1.3), we choose the basic function space as $L^1(\mathbb{R}^2)$ and we may derive the existence and uniqueness of the solution of the integral equation

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according to the results Giga-Miyakawa-Osada [15], Giga-Kambe [16]. The space $L^1(\mathbb{R}^2)$ is the invariant space under the scaling scaling

$$\begin{cases} n_\lambda(t,x) = \lambda^2 n(\lambda^2 t, \lambda x), \\ p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x), \\ \psi_\lambda(t,x) = \psi(\lambda^2 t, \lambda x), \end{cases}$$

that keep the equation invariant. It is important to solve the equation in such an invariant class because we may employ so called the *Fujita-Kato Principle* for the semilinear equation.

We first recall how to obtain the solution in the basic class $L^1(\mathbb{R}^2)$ following the results [15] and [16]. The similar argument is possible to apply for the system scaling (1.1) and (1.2) and for simplicity, we only treat the case (1.2).

We first introduce the corresponding integral equation for the Keller-Segel system (1.2);

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\nabla(u\nabla\psi)ds.$$

Let $\|u\| \equiv \sup_{t\in I} t^{1-1/p}\|u(t)\|_p$, where $I = [0,T)$ and $4/3 \le p < 2$. Then we have

$$\begin{aligned} \|u(t)\|_p \le & \|e^{t\Delta}u_0\|_p + C\int_0^t |t-s|^{-(1/r-1/p)-1/2}\|u\nabla\psi\|_r ds \\ \le & \|e^{t\Delta}u_0\|_p + C\int_0^t |t-s|^{-(1/r-1/p)-1/2}\|u(s)\|_p\|\nabla\psi\|_s ds \\ \le & \|e^{t\Delta}u_0\|_p + C\int_0^t |t-s|^{-1/p}\|u(s)\|_p^2 ds \\ \le & \|e^{t\Delta}u_0\|_p + C\int_0^t |t-s|^{-1/p}s^{-2(1-1/p)}ds\big(\sup_{t\in I} t^{(1-1/p)}\|u(t)\|_p\big)^2, \end{aligned} \tag{1.4}$$

where $1/r = 1/p + 1/s$ and $1/s = 1/p - 1/2$. Namely we have $1/r - 1/p + 1/2 = 1/p$. Then we need

$$\frac{1}{r} = \frac{2}{p} - \frac{1}{2} \le 1$$

and

$$\frac{2}{p} \le \frac{3}{2}$$

implies $4/3 \le p$. Hence under the condition $4/3 \le p < 2$, the integral

$$\int_0^t |t-s|^{-1/p}s^{-2(1-1/p)}ds = t^{(1-1/p)}B$$

converges, where $B > 0$ is a constant determined by the beta function. Then we have

$$\sup_I t^{1-1/p}\|u(t)\|_p \le \sup_I t^{1-1/p}\|e^{t\Delta}u_0\|_p + B\big(\sup_{t\in I} t^{(1-1/p)}\|u(t)\|_p\big)^2.$$

Now we see by $t \to 0$ the first term $\sup_I t^{1-1/p}\|e^{t\Delta}u_0\|_p$ can be small (this follows from the fact C_0^∞ is dense L^p and the initial data u_0 may be approximated by a C_0^∞ function that as $t \to 0$ we have the a priori bound for the solution. Analogous method may implies the estimate for the difference of the solutions. To show that the solution is belonging to $L^1(\mathbb{R}^2)$, we treat the integral equation in L^1 and use the L^1-L^1 boundedness for the heat flow to have

$$\begin{aligned}
\|u(t)\|_1 \le&\|e^{t\Delta}u_0\|_1 + C\int_0^t |t-s|^{-1/2}\|u\nabla\psi\|_1 ds \\
\le&\|u_0\|_1 + C\int_0^t |t-s|^{-1/2}\|u(s)\|_{4/3}\|\nabla\psi\|_4 ds \\
\le&\|u_0\|_1 + C\int_0^t |t-s|^{-1/2}\|u(s)\|_{4/3}\|u(s)\|_{4/3} ds \\
\le&\|u_0\|_1 + C\int_0^t |t-s|^{-1/2}s^{-1/2}ds\big(\sup_{t\in I} t^{1/4}\|u(t)\|_{4/3}\big)^2,
\end{aligned} \tag{1.5}$$

where

$$\|\nabla(-\Delta)^{-1}u(s)\|_4 \le C\|u(s)\|_{4/3}, \qquad \frac{1}{4}=\frac{3}{4}-\frac{1}{2}$$

is the Hardy-Littlewood-Sobolev inequality in $n=2$. The last integration is finite and it follows the uniform boundedness of $\sup_{t\in I} t^{1/4}\|u(t)\|_{4/3}$ and hence the solution belongs to L^1.

Giga-Miyakawa-Osada [15] applied the above method to the vortex equation (1.3) to show the existence and uniqueness of the solution for the initial data in $L^1(\mathbb{R}^2)$.

Proposition 1.1 (Giga-Miyakawa-Osada) *For $\omega_0 \in L^1(\mathbb{R}^2)$ the two dimensional vortex equation of the Navier-Stokes system* (1.3) *has a unique local solution $\omega(t) \in C\big([0,T);L^1(\mathbb{R}^2)\big) \cap C\big((0,T);L^p(\mathbb{R}^2)\big)$.*

We may apply the similar method to the 2 dimensional simplified Keller-Segel system and we may have the following:

Theorem 1.2 *For $u_0 \in L^1(\mathbb{R}^2)$, there exists a unique time local solution u for the two dimensional Keller-Segel equation* (1.2) *and it satisfies $u(t) \in C\big([0,T);L^1(\mathbb{R}^2)\big) \cap C\big((0,T);L^p(\mathbb{R}^2)\big)$.*

With this regards, the result in Kurokiba-Ogawa [24] can be extended into the case $1 \le p \le 2$ when we consider $n=2$.

For the vortex equation (1.3), the solution $\omega(t)$ satisfies the maximum principle and the uniform a priori bound follows. This shows the solution globally exists. On the other hand, for the Keller-Segel system (1.2) and the drift-diffusion system (1.1), the solutions of those system do not satisfy the maximum principle and hence we need to employ the entropy functional to establish the existence of the global solution. Indeed, Nagai-Senba-Yoshida [34], Biler [2], Nagai-Senba-Suzuki [33] employ the entropy functional to show the existence of the time global solution of (1.2) for a bounded domain Ω:

$$\begin{aligned}
\int_{\Omega|} u\log u dx - \frac{1}{2}\int_\Omega u\psi dx + \int_0^t\int_\Omega u\big|\nabla(\log u-\psi)\big|dxdt \\
\le \int_\Omega u_0\log u_0 dx - \frac{1}{2}\int_\Omega u_0(-\Delta)^{-1}u_0 dx.
\end{aligned} \tag{1.6}$$

We may derive the Lyapunov function from this entropy functional and the justification of this functional is very important. We see by a formal computation that

$$\int_\Omega u\log u dx - \frac{1}{2}\int_\Omega u\psi dx = \int_\Omega u\log u dx - \frac{1}{2}\|\nabla\psi\|_2^2,$$

however the second term of the right hand side does not have a sense as far as we consider the positive solution. Namely since $|x| \to \infty$

$$\psi \simeq \log |x|^{-1}$$

it follows that $\nabla\psi \notin L^2(\mathbb{R}^2)$.

This difficulty can be recovered by introducing "zero mean solutions". If we consider the case when the difference of the two carriers densities is equal to zero, we may consider the zero mean solution for the difference of the solution of the drift-diffusion system and one may justify the above integration by parts for this type of solutions. With this regards, we introduce the Hardy class $\mathcal{H}^1(\mathbb{R}^2)$ and then the entropy functional can have its meaning for the solution in this class.

Introducing $v = n + p$ and $w = n - p$, the equivalent form of the system (1.1) is as follows:

$$\begin{cases} \partial_t v - \Delta v + \nabla \cdot (w\nabla\psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\ \partial_t w - \Delta w + \nabla \cdot (v\nabla\psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\ -\Delta\psi = -\kappa w, \quad x \in \mathbb{R}^2, \\ \qquad v(0, x) = n_0(x) + p_0(x), \qquad w(0, x) = n_0(x) - p_0(x). \end{cases} \tag{1.7}$$

In this system we may naturally assume that w has the "zero mean value". For v, we consider the deviation from the average $v - \bar{v}$, where

$$\bar{v} = \int_{\mathbb{R}^2} v(t, x)dx = \int_{\mathbb{R}^2} v_0(x)dx$$

and we obtain the system:

$$\begin{cases} \partial_t v - \Delta v + \nabla \cdot (w\nabla\psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\ \partial_t w - \Delta w + \kappa\bar{v}w + \nabla \cdot (v\nabla\psi) = 0, \qquad t > 0, x \in \mathbb{R}^2, \\ -\Delta\psi = -\kappa w, \quad x \in \mathbb{R}^2, \\ \qquad v(t, x) \to 0, \quad w(t, x) \to 0, \quad |x| \to \infty, \\ \qquad v(0, x) = v_0 \equiv n_0(x) + p_0(x) - \overline{n_0 + p_0}, \\ \qquad w(0, x) = w_0 \equiv n_0(x) - p_0(x). \end{cases} \tag{1.8}$$

We choose the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ as the basic function class and shows the time local existence and well-posedness.

Definition. For $\lambda > 0$ and $\phi \in \mathcal{S}(\mathbb{R}^2)$, we let $\phi_\lambda = \lambda^{-2}\phi(\lambda^{-1}x)$ then for $0 < p < \infty$, we define the Hardy space $\mathcal{H}^p$ by

$$\mathcal{H}^p = \mathcal{H}^p(\mathbb{R}^2) = \left\{ f \in L^p_{loc}(\mathbb{R}^2); \quad \|f\|_{\mathcal{H}^p} \equiv \left\| \sup_{\lambda>0} |\phi_\lambda * f| \right\|_p < \infty \right\}.$$

In particular, it is well known that for $p = 1$, the dual space of the Hardy space $\mathcal{H}^1$ coincides with a class of the *bounded mean oscillation*)

$$BMO = \{f \in L^1_{loc}(\mathbb{R}^2); \|f\|_{BMO} < \infty\}.$$

Our main result is the following:

Theorem 1.3 (Ogawa-Shimizu[36]) *Let $\kappa = \pm 1$ and the initial data $(v_0, w_0) \in \mathcal{H}^1(\mathbb{R}^2) \times \mathcal{H}^1(\mathbb{R}^2)$ then there exists $T > 0$ and a unique solution (v, w) of (1.8) and satisfying v, $w \in C\big([0,T); \mathcal{H}^1\big) \cap L^2\big(0,T; \dot{\mathcal{H}}^{1,1}\big) \cap C\big((0,T); \dot{\mathcal{H}}^{2,1}\big) \cap C^1\big((0,T); \mathcal{H}^1\big)$. Moreover the solution flow map $(v_0, w_0) \to (v, w)$ is the Lipschitz continuous $\mathcal{H}^1(\mathbb{R}^2)^2 \to C\big([0,T); \mathcal{H}^1\big)^2$.*

The equation (1.8) has the invariant scale in the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and according to the Fujita-Kato principle, the existence of the time global solution in the same class for the small initial data immediately follows if the initial data is sufficiently small and $\kappa\bar{v} > 0$. However such a result is not quite important for this kind of system since the repulsive drift-diffusion system should have a global solution in large data and the attractive system (that has the opposite sign of the nonlinear coupling) has a finite time blowing up solution and it is required for identifying the threshold value for the global existence of the solution.

When $f \in \mathcal{H}^1(\mathbb{R}^2)$, it is known that the following estimate holds: For $f \in \mathcal{H}^1(\mathbb{R}^2)$, the Fourier transform $\hat{f}$ is subject to

$$\int_{\mathbb{R}^2} \frac{|\hat{f}(\xi)|^2}{|\xi|^2} d\xi \le C\|f\|_{\mathcal{H}^1}^2. \tag{1.9}$$

In this case, if $-\Delta\psi = w$ and $w \in \mathcal{H}^1(\mathbb{R}^2)$, then we have

$$\|\nabla\psi\|_2^2 = \int_{\mathbb{R}^2} |\xi\widehat{\psi}|^2 d\xi = \int_{\mathbb{R}^2} \frac{|\widehat{w}|^2}{|\xi|^2} d\xi \le C\|w\|_{\mathcal{H}^1}^2$$

and this shows the entropy functional remains finite for the solution in the Hardy space.

2 L^1 type energy inequality

The proof of Theorem 1.3 essentially relys on the endpoint type maximal regularity. The detailed proof can be found in [36]. We summarize the crucial part. For the solution of the initial value problem of the heat equation, the key estimate is considered as a type of the energy estimate: It is well known that the solution of heat equation

$$\begin{cases} \partial_t u - \Delta u = 0, & t > 0, x \in \mathbb{R}^n, \\ \quad u(0,x) = \phi(x). \end{cases} \tag{2.1}$$

satisfies the energy inequality:

$$\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds \le \|u_0\|_2^2$$

In particular, for the smooth solution we may derive the energy equality for the solution. The L^p version of the estimate with $1 < p < \infty$ is known as the parabolic estimate. We establish the corresponding estimate for $p = 1$ in the parabolic estimate when we exchange L^1 space into $\mathcal{H}^1$.

Theorem 2.1 *Let $e^{t\Delta}$ be the heat semi group and $\phi \in \mathcal{H}^1$. Then we have*

$$\left(\int_0^T \|\nabla e^{t\Delta}\phi\|_{\mathcal{H}^1}^2 dt\right)^{1/2} \le C\|\phi\|_{\mathcal{H}^1}, \tag{2.2}$$

where C is a positive constant independent of $T > 0$.

Theorem 2.2 *Let $1 < \theta < \infty$ and u be a solution of the inhomogeneous heat equation:*

$$\begin{cases} \partial_t u - \Delta u = f, & t > 0, x \in \mathbb{R}^n, \\ \quad u(0,x) = \phi(x). \end{cases} \tag{2.3}$$

Then we have

$$\|\nabla u\|_{L^2(I;\dot{\mathcal{B}}^0_{1,2})} \leq C \left(\|\phi\|_{\mathcal{H}^1} + \|f\|_{L^2(I;\mathcal{H}^{-1,1})} \right),$$

where $I = \mathbb{R}_+$.

This estimate is essentially equivalent to the endpoint type maximal regularity for the solution of the heat equation in the Besov space (cf. Ogawa-Shimizu [37]). The detailed proof requires the interpolation argument in the real analytic method.

The proof of Theorem 2.1 needs various estimates in the real and harmonic analysis and the meaning of the estimate is not directly understandable. To explain a heuristic reason that the estimate in Theorem 2.1 holds, we show the following proposition.

Proposition 2.3 *Let $e^{t\Delta}$ be the heat semi group and let $\phi \in L^1(\mathbb{R}^2)$. Then the following estimate*

$$\left(\int_0^T \|\nabla e^{t\Delta}\phi\|_1^2 dt \right)^{1/2} \leq C\|\phi\|_1 \tag{2.4}$$

generally fails.

Proof of Proposition 2.3. Noting the L^1-L^∞ estimate for the Fourier transform:

$$\sup_\xi |\hat{f}(\xi)| \leq \frac{1}{2\pi}\|f\|_1$$

we set $f = \nabla e^{t\Delta}\phi$ to see that

$$\begin{aligned}
\int_0^\infty \|\nabla e^{t\Delta}\phi\|_1^2 dt \geq & 4\pi^2 \int_0^\infty \left(\sup_\xi \left|\xi e^{-t|\xi|^2}\hat{\phi}(\xi)\right| \right)^2 dt \\
= & 4\pi^2 \int_0^\infty \frac{1}{t}\left(\sup_\xi \left|\sqrt{t}\xi e^{-t|\xi|^2}\hat{\phi}(\xi)\right| \right)^2 dt \\
& \text{(where we choose some } \eta \text{ instead of taking supremum over } \xi) \\
\geq & 4\pi^2 \int_0^\infty \frac{1}{t}\left(\left|\eta e^{-|\eta|^2}\right| \right)^2 |\hat{\phi}(\sqrt{t}^{-1}\eta)|^2 dt \\
\geq & C \int_0^\infty \frac{|\hat{\phi}(\sqrt{t}^{-1}\eta)|^2}{t} dt \\
& \text{letting} \frac{|\eta|}{\sqrt{t}} = r, \text{ we see from} \qquad dt = -2\frac{|\eta|^2}{r^3}dr \\
= & C \int_0^\infty \frac{|\hat{\phi}(r\omega)|^2}{r} dr.
\end{aligned}$$

Here $\omega = \eta/|\eta|$ is a unit vector. Integrating in $\omega \in \mathbb{S}^1$ and taking the average, we obtain that

$$\int_0^\infty \|\nabla e^{t\Delta}\phi\|_1^2 dt \ge C(\eta)\int_{\mathbb{R}^2} \frac{|\hat{\phi}(\xi)|^2}{|\xi|^2} d\xi. \tag{2.5}$$

By taking appropriate ϕ, the right hand side diverges in generally (for instance, choose $\hat{\phi}(0) \neq 0$). Even if the integral average of ϕ is 0, we may not obtain the finiteness of the right hand side only assuming that $\phi \in L^1(\mathbb{R}^2)$. ❑

We should note that Proposition 2.3 itself gives an another proof of the estimate (1.9). Note that Proposition 2.3 can be derived without using (1.9).

To establish the solvability of the system (1.1) in the Hardy space we need to show the bilinear estimate for the nonlinear term. The following estimate is a generalization of the well known estimate for the product of the divergence free vector and rotation free vector due to Coifman-Lions-Mayer-Semmes [8].

Proposition 2.4 *Let $\nabla w \in \mathcal{H}^1(\mathbb{R}^2)$ and $\nabla \psi \in \dot{H}^1 \cap L^\infty$. Then we have the following estimate:*

$$\|\nabla \cdot (w\nabla\psi)\|_{\mathcal{H}^1} \le C\big(\|w\|_2\|\Delta\psi\|_2 + \|\nabla w\|_{\mathcal{H}^1}\|\nabla\psi\|_\infty\big).$$

Analogous estimate of the above proposition is known in the Triebel-Lizorkin space, however it does not include the endpoint case: For $f,\ g \in \dot{F}^s_{p,\sigma} \cap L^r$ with $1 < p \le \infty$, $1 \le \sigma \le \infty$ $1/p = 1/r + 1/q$ and $s > 0$, it holds that

$$\|fg\|_{\dot{F}^s_{p,\sigma}} \le C\big(\|f\|_r\|g\|_{\dot{F}^s_{q,\sigma}} + \|f\|_{\dot{F}^s_{q,\sigma}}\|g\|_r\big)$$

The proof of the above estimate requires the L^p boundedness of the maximal function and the limiting case $p = 1$ is eliminated. Our estimate is corresponding to the case $p = 1$ by observing

$$\|\nabla(fg)\|_{\dot{F}^0_{1,2}} \simeq \|\nabla(fg)\|_{\mathcal{H}^1}.$$

3 Large time behavior of solutions of three dimensional critical drift-diffusion system

In this section, we consider the large time behavior of solutions for the drift diffusion system in three dimensions:

$$\begin{cases} \partial_t n - \Delta n + \nabla \cdot (n\nabla\psi) = 0, & t > 0, x \in \mathbb{R}^3, \\ \partial_t p - \Delta p - \nabla \cdot (p\nabla\psi) = 0, & t > 0, x \in \mathbb{R}^3, \\ -\Delta\psi = p - n, \quad x \in \mathbb{R}^3, \\ \quad n(0,x) = n_0(x), \qquad p(0,x) = p_0(x). \end{cases} \tag{3.1}$$

Carpio [6] obtained the large time asymptotic profile of the solution to the two and three dimensional Navier-Stokes system and it shows that the solutions can be expressed by the heat kernel and its derivatives as $t \to \infty$. This simultaneously shows the lower bound of the decaying solution. One may generalized this fact to the drift-diffusion system (3.1). For the parabolic Keller-Segel system, Nagai-Yamada [35] and M.Kato [21] have already showed that the asymptotic behavior of the large time solution.

Here we concentrate an analogous result for the drift-diffusion system in the case when $n = 3$. The solution of the initial value problem has an asymptotic expansion up to the second asymtotics. We consider the corresponding integral equation:

$$\begin{cases} n(t) = e^{t\Delta} n_0 - \displaystyle\int_0^t e^{(t-s)\Delta} \nabla \cdot (n(s)\nabla\psi(s))ds, \\ p(t) = e^{t\Delta} p_0 + \displaystyle\int_0^t e^{(t-s)\Delta} \nabla \cdot (p(s)\nabla\psi(s))ds, \\ \qquad - \Delta\psi = p - n. \end{cases}$$

If the decay of the nonlinear coupling in the above equation is faster than the decay order of the term of the initial data, then the asymptotic behavior of the solution is determined by the first term of the right hand sides. Indeed, the effect from the nonlinear term decays faster than the top term $G(t)$ since the each nonlinear coupling behaves as the square of the heat kernel in principle. Namely if we assume the decay of the initial data at $|x| \to \infty$, then we have for $1 \le q \le \infty$

$$\|n(t) - M_n G(t)\|_q = o(t^{-\gamma}), \quad \|p(t) - M_p G(t)\|_q = o(t^{-\gamma}), \qquad \gamma = \frac{n}{2}\left(1 - \frac{1}{q}\right), \tag{3.2}$$

where $M_n = \int_{\mathbb{R}^3} n(t)dx$, $M_p = \int_{\mathbb{R}^3} p(t)dx$ are total charges and under the positivety conditions $n_0 \ge 0$, $p_0 \ge 0$, those quantities are preserved and hence they are constants in time. Now our main concern here is the second asymptotic profiles. For $n = 3$, the second expansion of the large time solution meets the critical situations.

Using the first asymptotic expansion (3.2) and noting the nonlinear term decays faster in t, then one may expect that the second asymptotic would be obtained by the derivatives of the heat kernel. From $\|n(t)\|_q = O(t^{-\gamma})$ $(\gamma = \frac{n}{2}(1 - \frac{1}{q}))$ if the coefficient of the nonlinear coupling

$$V_n = \int_0^\infty \int_{\mathbb{R}^n} n(s,x)\nabla(-\Delta)^{-1}(n(s,x) - p(s,x))dxds$$

is well-defined, then the inner product of V_n (or V_p) and $\nabla G(t)$ may be the candidate of the second asymptotic form. This was observed by Escobedo-Zuazua [9] for the solution of the heat convention equation and for the solution of the Navier-Stokes system by Carpio [6] in lower dimensions $n = 2, 3$, and general dimensions by Fujigaki-Miyakawa [10]. When $n = 3$, we have $\|n(s)\|_\infty = O(s^{-3/2})$ and $\|\nabla(-\Delta)^{-1}(n(s) - p(s))\|_q = O(s^{-\gamma+\frac{1}{2}})$ thus if $q = 1$ we see formally that

$$V = \int_0^\infty (1+s)^{-1} ds = \infty$$

and the coefficient of the asymptotic may possibly diverge. This is not the case for the asymptotic behavior of the solution of the Navier-Stokes equation or the parabolic Keller-Segel system (see Nagai-Yamada [35]);

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u\nabla\psi) = 0, \qquad t > 0, x \in \mathbb{R}^n, \\ \partial_t \psi - \Delta\psi + \psi = u, \qquad t > 0, x \in \mathbb{R}^n, \\ \qquad u(0,x) = u_0(x), \qquad \psi(0,x) = \psi_0(x) \end{cases} \tag{3.3}$$

under the condition $n \geq 2$.

We then introduce the correction term for the asymptotic behavior and obtain the second asymptotic expansion of the solution for (3.1) in $n = 3$.

Theorem 3.1 (Ogawa-Yamamoto [38]) *Let $\kappa = 1$ and $3/2 \leq r < 3$. For the initial data $(n_0, p_0) \in \big(L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\big)^2$ with $n_0 \geq 0$, $p_0 \geq 0$, there exists a unique global solution (n, p) to (3.1) and for $1 \leq q \leq \infty$, n, $p \in C\big([0,\infty); L^q\big) \cap C^1\big((0,\infty); L^r\big) \cap C\big((0,\infty); W^{2,r}\big)$. Moreover under the extra condition $n_0, p_0 \in L^1_2(\mathbb{R}^3)$, the solution satisfies the following asymptotic behavior: For $1 \leq q \leq \infty$ and $G(t) = \big(\frac{1}{4\pi t}\big)^{3/2} \exp(-\frac{|x|^2}{4t})$, we have*

$$\begin{aligned} \|n(t) - M_n G(t) - (m_n + V_n) \cdot \nabla G(t) + M_n W J(t)\|_q &= o(t^{-\gamma - \frac{1}{2}}), \quad t \to \infty, \\ \|p(t) - M_p G(t) - (m_p + V_p) \cdot \nabla G(t) + M_p W J(t)\|_q &= o(t^{-\gamma - \frac{1}{2}}), \quad t \to \infty, \end{aligned}$$

where $\gamma = \frac{3}{2}\left(1 - \frac{1}{q}\right)$. The correction term $J(t)$ is given by the heat kernel $G(t)$ as

$$J(t) = \int_0^t \nabla G(t-s) * \big(G(1+s)\nabla(-\Delta)^{-1} G(1+s)\big) ds$$

and $W = M_n - M_p$, $m_n = \int_{\mathbb{R}^3} x n_0(x) dx$, $m_p = \int_{\mathbb{R}^3} x p_0(x) dx$,

$$\begin{cases} V_n = \displaystyle\int_0^\infty \int_{\mathbb{R}^3} n(s) \nabla(-\Delta)^{-1}(n(s) - p(s)) dx ds, \\ V_p = \displaystyle\int_0^\infty \int_{\mathbb{R}^3} p(s) \nabla(-\Delta)^{-1}(n(s) - p(s)) dx ds. \end{cases}$$

The constant vectors V_p and V_n seem diverge in a rough observation, however those quantities are finite and well-defined indeed. This means that the problem is rather similar to the case when we consider the asymptotic behavior of the solution in the Navier-Stokes system. When the initial data satisfies the electrically equilibrium condition $W = 0$, then the result is reduced into the same situation to the higher dimensions.

We should notice that recently Yamada [43] obtained the higher order asymptotic expansion for the solution of the parabolic Keller-Segel system (3.3) up to order n. The main difference between the system of (3.3) and (3.1) is that the principal symbol appearing in front of the nonlinear coupling is not smooth for (3.1) while the former case, it is smooth. Hence the large time asymptotic can be determined by essentially the heat kernel and its combinations for the parabolic Keller-Segel case. Indeed the correction term $J(t)$ for the solution of (3.1) can be obtained similarly but slightly simpler form such as

$$J(t) = \int_0^t G(t-s) * \Delta G^2(1+s) ds.$$

One can show easily that

$$J(t) = \int_0^t \left(\frac{1}{4\pi(1+s)}\right)^{n/2} \Delta G\left(t - \frac{s-1}{2}\right) ds$$

and it follows that $J(t) \simeq (1+t)^{-n}$ (cf. [21]). For the drift-diffusion case, however, the inverse of the Laplacian involves a singularity and the symbol of the operator appears in the nonlinear term is not smooth. This makes the problem slightly complicated since the solution does not have the fast decaying properties in space direction. This is connecting the basic nature of the solution to (3.1) that has a stronger non-local effect than the parabolic Keller-Segel system (3.3). In our case, we may show that

$$C_1(1+t)^{-\gamma-1/2} \le \|J(t)\|_q \le C_2(1+t)^{-\gamma-1/2}.$$

The upper bound can be obtained by noting that

$$\int_0^t \nabla G(t-s) \int_{\mathbb{R}^3} G(s)\nabla(-\Delta)^{-1}G(s)dxds = 0.$$

Finaly, we show the upper and lower bound for $\|J(t)\|_\infty$. From the above we see from

$$\int_0^t \nabla G(t-s) \int_{\mathbb{R}^3} G(1+s)\nabla(-\Delta)^{-1}G(1+s)dxds = 0.$$

that

$$\begin{aligned}
\|J(t)\|_\infty =& \left\| J(t) - \int_0^t \nabla G(t-s) \int_{\mathbb{R}^3} G(s)\nabla(-\Delta)^{-1}G(s)dxds \right\|_\infty \\
\le& \Big\| \int_0^t \int_{\mathbb{R}^3} \big(\nabla G(t-s,x-y) - \nabla G(t-s,x)\big) \\
& \qquad \times G(1+s,y)\nabla(-\Delta)^{-1}G(1+s,y)dxds \Big\|_\infty \\
\le& C(1+t)^{-2}.
\end{aligned}$$

While for the lower bound, we employ the Plancherel identity to see that

$$\begin{aligned}
J(t)|_{x=0} =& \int_0^t \int_{\mathbb{R}^3} e^{-(t-s)|\xi|^2}(i\xi)\cdot \int_{\mathbb{R}^3}(-i)\frac{\eta}{|\eta|^2} e^{-(1+s)|\eta-\xi|^2} e^{-(1+s)|\eta|^2} d\eta d\xi ds \\
=& \int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\xi\cdot\eta}{|\eta|^2} e^{-(1+t)|\xi-\sigma(t,s)\eta|^2} e^{-(2t+1-s)\sigma(t,s)|\eta|^2} d\eta d\xi ds,
\end{aligned} \tag{3.4}$$

where

$$\sigma(t,s) = \frac{1+s}{1+t}.$$

Since the odd integrant is vanishing, we see by the change of variable that

$$\begin{aligned}
J(t)|_{x=0} =& \int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{(\zeta+\sigma(t,s)\eta)\cdot\eta}{|\eta|^2} e^{-(1+t)|\zeta|^2} e^{-(1+2t-s)\sigma(t,s)|\eta|^2} d\eta d\zeta ds \\
=& \int_0^t \sigma(t,s) \iint_{\mathbb{R}^3\times\mathbb{R}^3} e^{-(1+t)|\zeta|^2} e^{-(1+2t-s)\sigma(t,s)|\eta|^2} d\eta d\zeta ds \\
=& \pi^3 \left(\frac{1}{1+t}\right)^2 \int_0^t (1+s)^{1/2} \left(\frac{1}{1+2t-s}\right)^{3/2} ds
\end{aligned} \tag{3.5}$$

which shows the lower bound of $J(t)|_{x=0}$ as $(1+t)^{-2}$.

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