Large time behaviour of solutions of semilinear heat equations in $\mathbb{R}^N$

Hiroki Nagatomo
(Tokyo University of Science)

We consider the large time behaviour of solutions to the Cauchy problem for semilinear heat equations of the form

$$(\mathbf{CP}) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + f(u) = 0 & \text{in } \mathbb{R}^N \times (0,\infty), \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

where u_0 is a real-valued initial data.

Gmira-Veron [**1**] have treated the large time behaviour of solutions to (**CP**) in the typical case $f(s) = s|s|^{p-1}$, $p > 1$. Especially, it was proved that the behaviour depends strongly on the critical value $p_N := 1 + 2/N$ and on the rate of decay of the initial data u_0 as $|x| \to \infty$. Later, Kajikiya [**2**] considered the more general semilinear term such that $f \in C(\mathbb{R})$, f is nondecreasing, $sf(s) > 0$ for $s \neq 0$, and f satisfies one of the following conditions near $s = 0$:

(C.1) $\displaystyle \overline{\lim_{s\to 0}} \frac{|f(s)|}{|s|^{1+2/N}} = 0$ (if $f(s) = s|s|^{p-1}$, then $p > 1 + 2/N$);

(C.2) $\displaystyle \lim_{s\to 0} \frac{|f(s)|}{|s|^{1+2/N}} =: K \in (0,\infty)$ (if $f(s) = s|s|^{p-1}$, then $p = 1 + 2/N$);

(C.3) $\displaystyle 0 < K_1 := \underline{\lim_{s\downarrow 0}} \frac{f(s)}{s^p} \leq \overline{\lim_{s\downarrow 0}} \frac{f(s)}{s^p} =: K_2 < \infty$ for some $1 < p < 1 + 2/N$ (and hence $\displaystyle \lim_{s\to 0} \frac{|f(s)|}{|s|^{1+2/N}} = \infty$) and $f(s)/s$ is nondecreasing on $(0, \eta)$ for some $\eta > 0$.

The purpose of this talk is to explain [**2**] with some modifications.

Theorem. *Let u be a solution to* (**CP**) *with $u_0 \in L^1(\mathbb{R}^N)$. Put $E_c(t) := \{x \in \mathbb{R}^N; |x| \leq c\sqrt{t}\}$.*

(1) *Suppose condition* (C.1). *Then for any $c > 0$,*

$$\lim_{t\to\infty} \sup_{x\in E_c(t)} |t^{N/2}u(x,t) - C_0(4\pi)^{-N/2}\exp(-|x|^2/4t)| = 0,$$

where $\displaystyle C_0 := \lim_{t\to\infty} \int_{\mathbb{R}^N} u(x,t)dx = \int_{\mathbb{R}^N} u_0(x)dx - \int_0^\infty \int_{\mathbb{R}^N} f(u(x,t))dxdt.$

(2) *Suppose condition* (C.2). *Then for any $c > 0$,*

$$\lim_{t\to\infty} \sup_{x\in E_c(t)} |t^{N/2}u(x,t)| = 0.$$

(3) *Suppose condition* (C.3). *Assume further that $u_0 \geq 0$ and for any $A > 0$ there exists $R > 0$ such that $u_0(x) \geq A|x|^{-2/(p-1)}$ ($|x| \geq R$). Then for any $c > 0$,*

$$\left(\frac{\gamma}{K_2}\right)^\gamma \leq \underline{\lim_{t\to\infty}} \left[\inf_{x\in E_c(t)} t^\gamma u(x,t)\right] \leq \overline{\lim_{t\to\infty}} \left[\sup_{x\in E_c(t)} t^\gamma u(x,t)\right] \leq \left(\frac{\gamma}{K_1}\right)^\gamma,$$

where $\gamma := 1/(p-1)$.

References

[1] A. Gmira, L. Veron, *Large time behaviour of the solutions of a semilinear parabolic equation in* $\mathbb{R}^N$, J. Differential Equations **53** (1984), 258–276.

[2] R. Kajikiya, *On the asymptotic behavior of solutions of certain semilinear parabolic equations in* $\mathbb{R}^N$, Hiroshima Math. J. **16** (1986), 85–99.