Parabolic Schrödinger operators with potentials which belong to the parabolic reverse Hölder class

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Let $0 \le V \in L^p_{\mathrm{loc}}$ $(1 < p < \infty)$. Then we consider the parabolic Schrödinger operator

$$\mathcal{A} := \partial_t - \Delta + V \quad \text{in} \quad L^p(\mathbb{R}^{N+1}),$$

and define $\mathcal{A}_p := \mathcal{A}$ with maximal domain $D(\mathcal{A}_p) := \{u \in L^p;\ Vu \in L^1_{\mathrm{loc}},\ (\partial_t - \Delta + V)u \in L^p\}$. Then the parabolic Kato's inequality implies that $\mathcal{A}_p$ is m-accretive stated in [**1**].

If we have a separation property;

$$\|(\partial_t - \Delta)u\|_p + \|Vu\|_p \le M\|(\lambda + \partial_t - \Delta + V)u\|_p \quad \forall u \in D(\mathcal{A}_p),$$

then we obtain the domain characterization. To show it we restrict $V = V(x,t)$ to the parabolic reverse Hölder class $(PRH)_p$ for some $p > 1$.

To define the class $(PRH)_p$ we denote by $K(X_0, R)$ the parabolic cylinder of center $X_0 = (x_0, t_0) \in \mathbb{R}^N \times \mathbb{R}$ and radius $R > 0$:

$$K(X_0, R) := \{X = (x,t) \in \mathbb{R}^N \times \mathbb{R};\ |x - x_0| < R,\ |t - t_0| < R^2\}.$$

Definition. Let $1 < p < \infty$. We say that $V \in (PRH)_p$ if $V \in L^p_{\mathrm{loc}}$, $V > 0$ a.e. and there exists a positive constant $C = C(p, V)$ such that

$$(*) \qquad \left(\frac{1}{|K|}\int_K V(x,t)^p\,dxdt\right)^{\frac{1}{p}} \le \frac{C}{|K|}\int_K V(x,t)\,dxdt$$

for every parabolic cylinder K. Define C_0 as the smallest positive constant C in $(*)$.

The purpose of this talk is to establish the following proposition (for part (a) cf. [**2**]).

Proposition. *Assume that $V \in (PRH)_p$. Then*

(a) *there exists $\delta = \delta(C_0) > 0$ such that $V \in (PRH)_{p+\delta}$;*

(b) *there exists $1 \le s < \infty$ and $c = c(C_0) > 0$ such that*

$$\left(\frac{1}{|K|}\int_K g\right)^s \le \frac{c}{V(K)}\int_K g^s V$$

for nonnegative functions $g \in L^{sp'}_{\mathrm{loc}}$ and parabolic cylinders K, where $V(K) = \int_K V$.

Based on this proposition we can prove the domain characterization stated in [**1**].

Theorem. *Let $1 < p < \infty$. If $V \in (PRH)_p$, then there exists a positive constant $M_i = M_i(p, C_0)$ $(i = 1, 2)$ such that*

$$\|Vu\|_p \le M_1\|(\lambda + A_p)u\|_p, \quad \|(\partial_t - \Delta)u\|_p \le M_2\|(\lambda + A_p)u\|_p$$

for all $u \in D(\mathcal{A}_p)$. Consequently

$$D_p(\mathcal{A}) = W_V^{(2,1),p} := \{u \in L^p;\ \partial_t u, D_x^2 u, Vu \in L^p\},$$

i.e., the solution u of $\partial_t u - \Delta u + Vu + \lambda u = f$ $(\lambda > 0,\ f \in L^p)$ belongs to $W_V^{(2,1),p}$.

References

[1] A. Carbonaro, G. Metafune, C. Spina, *Parabolic Schrödinger operators*, J.Math.Anal.Appl., **343** (2008), 965–974.

[2] F. W. Gehring, *The L^p-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math., **130** (1973), 265–277.