Analytic C_0-semigroups on $L^p(\mathbb{R}^N)$ generated by second order elliptic operators with unbounded coefficients

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In this talk we consider the analytic C_0-semigroup on $L^p(\mathbb{R}^N)$ $(1 < p < \infty)$ generated by $-A_p$, where A_p is the following second order elliptic operator:

$$A_p u := -\mathrm{div}(a\nabla u) - F\cdot\nabla u + Vu = -\sum_{j,k=1}^{N}\frac{\partial}{\partial x_k}\left(a_{jk}\frac{\partial u}{\partial x_j}\right) - \sum_{j=1}^{N} F_j\frac{\partial u}{\partial x_j} + Vu$$

for $u \in D(A_p) := \big\{u \in W^{2,p}(\mathbb{R}^N) \,|\, Vu \in L^p(\mathbb{R}^N)\big\}$.

Here the coefficients $a = (a_{jk}), F = (F_j)$ and V satisfy the following conditions:

(H0) $a \in C^1(\mathbb{R}^N;\mathbb{R}^{N\times N}) \cap W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^{N\times N})$, $F \in C^1(\mathbb{R}^N;\mathbb{R}^N)$, $0 \le V \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$;
(H1) $a_{jk} = a_{kj}$ and $a_0[\xi] := \sum_{j,k=1}^N a_{jk}(x)\xi_j\xi_k \ge \nu|\xi|^2$ $(x,\xi \in \mathbb{R}^N)$ for some constant $\nu > 0$;
(H2) $c_0 \le U \le V \le c_1 U$ for some constants $c_0, c_1 > 0$;
(H3) $|F\cdot\xi| \le \kappa U^{1/2} a_0[\xi]^{1/2}$ for some constant $\kappa \ge 0$;
(H4) $\mathrm{div}F + \theta U \ge 0$ for some constant $0 \le \theta < 1$,
where $U \in C^1(\mathbb{R}^N;\mathbb{R})$ satisfies that $a_0[\nabla U]^{1/2} \le \gamma U^{3/2} + C_\gamma$ for some $\gamma > 0$ and $C_\gamma \ge 0$.

Metafune-Pallara-Prüss-Schnaubelt [**1**] showed that, under almost the same conditions, $-A_p$ generates a C_0-semigroup on $L^p(\mathbb{R}^N)$ analytic and contractive in the sector $\Sigma(\pi/2 - \tan^{-1}\delta_p)$, where $\Sigma(\psi) := \{z \in \mathbb{C}\,;\, |\arg z| < \psi\}$ and

$$\delta_p^2 := \frac{(p-2)^2}{4(p-1)} + \frac{\kappa^2}{4(1-\theta/p)}.$$

In particular, if $F \equiv 0$ and $V \equiv 0$, then $\kappa = \theta = 0$ in conditions **(H3)**, **(H4)** and hence δ_p coincides with the known constant $|p-2|/(2\sqrt{p-1})$ determined by Okazawa [**2**].

The purpose of this talk is to show that $-A_p$ generates a C_0-semigroup on $L^p(\mathbb{R}^N)$ analytic and possibly non-contractive in the extended sector $\Sigma(\pi/2 - \tan^{-1}\delta)$, where $\delta := \min\{\delta_2, \delta_p\}$.

Main Theorem. *Let $1 < p < \infty$. Assume that conditions* **(H0)**–**(H4)** *are satisfied with*

$$\frac{\theta}{p} + (p-1)\gamma\left(\frac{\kappa}{p} + \frac{\gamma}{4}\right) < 1.$$

Then $-A_p$ generates a C_0-semigroup on $L^p(\mathbb{R}^N)$ analytic in the sector $\Sigma(\pi/2 - \tan^{-1}\delta)$, where $\delta := \min\{\delta_2, \delta_p\}$.

The proof is carried out by generalizing the exposition in Ouhabaz [**3**].

Remark. If $p \in (1,2) \cup (p_0, \infty)$ (for some $p_0 > 2$), then we can show that $\delta_2 < \delta_p$, that is, our result improves the previous result in [**1**].

References

[1] G. Metafune, D. Pallara, J. Prüss, R. Schnaubelt, *L^p-theory for elliptic operators on $\mathbb{R}^d$ with singular coefficients*, Z. Anal. Anwendungen **24** (2005), 497–521.

[2] N. Okazawa, *Sectorialness of second order elliptic operators in divergence form*, Proc. Amer. Math. Soc. **113** (1991), 701–706.

[3] E. M. Ouhabaz, "*Analysis of Heat Equations on Domains*," Princeton Univ. Press, Oxford, 2005.