Cauchy problem for nonlinear Schrödinger equations with inverse-square potentials

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In this talk we consider the following Cauchy problem for nonlinear Schrödinger equations with inverse-square potentials

$$(\mathbf{NLS}) \qquad \begin{cases} i\dfrac{\partial u}{\partial t} = -\Delta u + \dfrac{a}{|x|^2}u + f(u) & \text{in } \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = u_0(x) & \text{on } \mathbb{R}^n, \end{cases}$$

where $i=\sqrt{-1}$, $n \geq 2$, $a > -\frac{(n-2)^2}{4}$ and $f : \mathbb{C} \to \mathbb{C}$ is a nonlinear function satisfying
(**N1**) $f \in C^1(\mathbb{R}^2;\mathbb{R}^2)$ and $f(0)=0$;
(**N2**) $|f(u)-f(v)| \leq K(1+|u|+|v|)^{p-1}|u-v|$ $(u, v \in \mathbb{C})$ for some $K \geq 0, p > 1$;
(**N3**) $f(x) \in \mathbb{R}$ $(x>0)$ and $f(e^{i\theta}z) = e^{i\theta}f(z)$ $(z \in \mathbb{C}, \theta \in \mathbb{R})$;
(**N4**) $F(x) := \int_0^x f(s)ds \geq -L_1x^2 - L_2x^{q+1}$ $(x>0)$ for some $L_1, L_2 \geq 0$ and $1 < q < 1+\frac{4}{n}$.
For example, $f(u) := |u|^{p-1}u$ satisfies (**N1**)–(**N4**) for $p>1$ and $f(u) := -|u|^{p-1}u$ satisfies (**N1**)–(**N4**) for $1<p<1+\frac{4}{n}$.

A function $u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$ is said to be a *weak solution* to (**NLS**) if u satisfies (**NLS**) in the sense of $H^{-1}(\mathbb{R}^n)$ and belongs to $C(\mathbb{R};H^1(\mathbb{R}^n)) \cap C^1(\mathbb{R};H^{-1}(\mathbb{R}^n))$. (**NLS**) is *well-posed* if there exists a unique weak solution for every $u_0 \in H^1(\mathbb{R}^n)$. Well-posedness for (**NLS**) is well-known when $a=0$ (see [**2, 3**]). The purpose of this talk is to show it when $a \neq 0$. Based on the Strichartz estimates for $S_a(t) := e^{-it(-\Delta + a|x|^{-2})}$ established in [**1**], we can apply the contraction principle to the integral equation associated with (**NLS**):

$$(\mathbf{INT}) \qquad u(t) = S_a(t)u_0 - i\int_0^t S_a(t-s)f(u(s))ds$$

$$\text{in } X_T := \left\{ \begin{array}{ll} v \in L^\infty(-T,T;H^1(\mathbb{R}^n)); & \nabla v \in L^r(-T,T;L^{p+1}(\mathbb{R}^n)), \\ & \|v\|_{L_t^\infty L_x^2}, \|\nabla v\|_{L_t^\infty L_x^2}, \|\nabla v\|_{L_t^r L_x^{p+1}} \leq 2\eta \end{array} \right\},$$

where $r := \frac{4(p+1)}{n(p-1)}$. Then we obtain the following well-posedness for (**NLS**) when $a \neq 0$.

Main Theorem. Assume that $n \geq 3$, $1 < p < \frac{n+2}{n-2}$, $a > \left[\frac{n(p-1)}{2(p+1)}\right]^2 - \frac{(n-2)^2}{4}$ and (**N1**)–(**N4**). Then (**NLS**) is well-posed, that is, for every $u_0 \in H^1(\mathbb{R}^n)$ there exists a unique weak solution to (**NLS**). Moreover u satisfies $\|u(t)\|_{L_x^2} = \|u_0\|_{L_x^2}$ and $E(u(t)) = E(u_0)$ for $t \in \mathbb{R}$, where

$$E(v) := \frac{1}{2}\|\nabla v\|_{L_x^2}^2 + \frac{a}{2}\left\|\frac{v}{|x|}\right\|_{L_x^2}^2 + \int_{\mathbb{R}^n} F(|v|)dx.$$

Remark. We also construct a weaker solution $u \in C(\mathbb{R};L^2(\mathbb{R}^n))$ to (**INT**) with $u_0 \in L^2(\mathbb{R}^n)$.

References

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