L^∞-estimates for evolution equations with p-Laplacian

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Let Ω be a bounded domain in $\mathbb{R}^N (N \in \mathbb{N})$ with smooth boundary $\partial\Omega$. In this talk we consider the following initial-boundary value problem:

$$(\mathbf{E})_p \qquad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) \in \Delta_p u(x,t) - g(u(x,t)) + h(x), & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Here u is a real-valued unknown function, $\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u)$, $\max\{1, \frac{2N}{N+2}\} < p < \infty$, $h : \Omega \to \mathbb{R}$ is a given function, and $g : \mathbb{R} \to 2^{\mathbb{R}}$ is a (possibly, multi-valued) function satisfying

(A1) there exist functions $g_0 : \mathbb{R} \to 2^{\mathbb{R}}$ and $g_1 : \mathbb{R} \to \mathbb{R}$ such that $g(\xi) = g_0(\xi) + g_1(\xi)$ $(\forall\, \xi \in \mathbb{R})$,

$(\eta_1 - \eta_2)(\xi_1 - \xi_2) \geq 0$ $(\forall\, \xi_j \in \mathbb{R}\ \forall\, \eta_j \in g_0(\xi_j), j = 1, 2)$,

$\xi \mapsto \xi + g_0(\xi)$ is surjective from $\mathbb{R}$ to $\mathbb{R}$, and $0 \in g_0(0)$,

$|g_1(\xi) - g_1(\eta)| \leq L|\xi - \eta|$ $(\forall\, \xi, \eta \in \mathbb{R})$ for some $L > 0$, and $g_1(0) = 0$;

(A2) if $p \leq 2$, then there exist constants θ, k_0, $k_1 > 0$ such that for $\xi \in \mathbb{R}$,

$$|\overset{\circ}{g_0}(\xi)| \geq k_0|\xi|^{1+\theta} - k_1,$$

where $\overset{\circ}{g_0}(\xi)$ denotes the minimal section of $g_0(\xi)$, that is, $\overset{\circ}{g_0}(\xi) := \mathrm{Proj}_{g_0(\xi)}0$.

Efendiev-Ôtani [**2**] showed that, under condition **(A1)**, for every h, $u_0 \in L^2(\Omega)$ there exists a unique solution $u \in C([0,\infty); L^2(\Omega)) \cap W^{1,2}_{\mathrm{loc}}((0,\infty); L^2(\Omega)) \cap L^p_{\mathrm{loc}}([0,\infty); W^{1,p}_0(\Omega))$ to $(\mathbf{E})_p$. They also obtained L^2- and L^∞-estimates of the solution under conditions **(A1)**, **(A2)** when $h \in L^\infty(\Omega)$. The proof of the L^∞-estimate in [**2**] needs an L^δ-estimate $(\delta > 2)$ and depends strongly on the result for generalized quasilinear equations established by DiBenedetto [**1**, Theorem V.3.2] of which the statement and proof are not so simple.

The purpose of this talk is to give a simplified proof of the L^∞-estimate without using L^δ-estimates $(\delta > 2)$ and obtain the detailed information about the time decay. Combining the L^2-estimate with the argument in Takeuchi-Yokota [**3**], we can obtain the following theorem.

Main Theorem. *Let $p > \max\{1, \frac{2N}{N+2}\}$ and $h \in L^\infty(\Omega)$. Assume that conditions* **(A1)**, **(A2)** *are satisfied. Let u be the unique solution to $(\mathbf{E})_p$ with $u_0 \in L^2(\Omega)$. Then there exist positive constants $c_j = c_j(p, N, |\Omega|, \|h\|_{L^\infty(\Omega)})$ $(j = 1, 2)$, independent of u_0, such that for every $t > 1$,*

$$\|u(t)\|_{L^\infty(\Omega)} \leq \max\{1, c_1 + c_2(t-1)^{-\frac{2p}{N(q-2)(\delta-2)}}\},$$

where $q = \frac{p(N+2)}{N}$, $\delta = 2 + \theta$ if $p \leq 2$, and $\delta = p$ if $p > 2$.

Remark. We can construct an infinite-dimensional attractor for $(\mathbf{E})_p$ by using the above-mentioned L^∞-estimate, while a $C^{1,\alpha}$-estimate is used in their construction in [**2**].

References

[1] E. DiBenedetto, *Degenerate Parabolic Equations*, Universitext, Springer-Verlag, New York, 1993.

[2] M.A. Efendiev, M. Ôtani, *Infinite-dimensional attractors for evolution equations with p-Laplacian and their Kolmogorov entropy*, Differential Integral Equations **20** (2007), 1201–1209.

[3] S. Takeuchi, T. Yokota, *Global attractors for a class of degenerate diffusion equations*, Electron. J. Differential Equations **2003** (2003), 1–13.