Existence and uniqueness for semilinear parabolic equations in Banach spaces

Kousuke Matsunaga
(Tokyo University of Science)

In this talk we consider the Cauchy problem for semilinear parabolic equations of the form

$$(\mathbf{CP})\qquad \begin{cases} u'(t) = Au(t) + J(u(t)), \\ u(0) = \varphi \end{cases}$$

in a Banach space E with norm $\|\cdot\|$, where $u : [0,T] \to E$ is an unknown function, A is the generator of a C_0-semigroup $\{e^{tA}\}_{t\geq 0}$ on E such that $\|e^{tA}\varphi\| \leq Me^{\gamma t}\|\varphi\|$ ($\varphi \in E,\ t \geq 0$) for some constants $M \geq 1$ and $\gamma \geq 0$, and J is a nonlinear function from a subset of E into E. Suppose that the domain of J is itself a Banach space E_J with norm $|\cdot|_J$ and $J : E_J \to E$ is locally Lipschitz continuous, i.e., for every $r > 0$ there exists $\ell(r) > 0$ such that

$$\|J\varphi - J\psi\| \leq \ell(r)|\varphi - \psi|_J, \quad \varphi,\ \psi \in E_J\ (|\varphi|_J \leq r,\ |\psi|_J \leq r).$$

Also we assume that there exist positive constants N and a with $a < 1$ such that

$$|e^{tA}\varphi|_J \leq Nt^{-a}\|\varphi\|,\ \varphi \in E,\ t \in (0,T].$$

Moreover, suppose that $e^{tA}\varphi \in C((0,T];E_J)$ for every $\varphi \in E$.

The first purpose is to give a detailed proof of the following existence theorem [**1**,Theorem 2] for "continuous" solutions to the corresponding integral equation

$$(\mathbf{IE})\qquad u(t) = e^{tA}\varphi + \int_0^t e^{(t-s)A}J(u(s))ds.$$

Theorem 1. *Let* $E,\ A,\ \{e^{tA}\}_{t\geq 0},\ J,\ E_J,\ \ell(r),\ N$ *and* a *be as above.*

a) *Assume that* $\int_\tau^\infty \ell(r)r^{-1/a}dr < \infty$ *for some* $\tau > 0$. *Then for every* $\varphi \in E$ *there exists a unique solution* $u \in C([0,T_\varphi);E) \cap C((0,T_\varphi);E_J)$ *to* (**IE**) *such that* $\overline{\lim}_{t\downarrow 0} t^a|u(t)|_J < \infty$, *where* $T_\varphi := \sup\{T;$ *a continuous solution to* (**IE**) *exists on* $[0,T]\}$. *Moreover, if* $T_\varphi < \infty$, *then* $\lim_{t\uparrow T_\varphi}\|u(t)\| = \infty$ *and* $\overline{\lim}_{t\uparrow T_\varphi}|u(t)|_J = \infty$.

b) *Assume that* $\int_1^\infty \ell(r)r^{-1/a}dr = \infty$ *and there exist constants* $C > 0$, $b \in (0,a)$ *and* $r_0 \geq 0$ *such that* $\ell(r) \leq Cr^{\frac{1-a}{b}}$ *for every* $r \geq r_0$. *Let* $K_0 > 0$ *with* $NBC(2K_0)^{\frac{1-a}{b}} \leq 1/2$ *and* $T_2 > 0$ *with* $Me^{\gamma T_2}C(2K_0)^{\frac{1-a}{b}}(a-b)^{-1}T_2^{a-b} < 1$ *and* $r_0{T_2}^b \leq 2K_0$, *where* $B := \int_0^1 (1-s)^{-a}s^{a-1-b}ds$. *If* $\varphi \in E$ *satisfies* $t^b|e^{tA}\varphi|_J \leq K_0$ *for every* $t \in (0,T_2]$, *then there exists a unique solution* $u \in C([0,T_2];E) \cap C((0,T_2];E_J)$ *to* (**IE**) *such that* $t^b|u(t)|_J \leq 2K_0$ *for every* $t \in (0,T_2]$.

The second purpose is to show the following corollary [**1**, Corollary 2.1] for the "differentiability" of the solution $u(t)$ in Theorem 1 by employing the argument in [**2**].

Corollary 2. *Assume that* $\{e^{tA}\}_{t\geq 0}$ *is an analytic semigroup on both* E *and* E_J. *Then the solution* $u(t)$ *in* Theorem 1 *is continuously differentiable on* $(0,T)$ *in* E, $u(t) \in D(A)$ *for every* $t \in (0,T)$ *and satisfies* (**CP**), *where* $T = T_\varphi$ *or* $T = T_2$.

References

[1] F.B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in* L^p, Indiana Univ. Math. J. **29** (1980), 79–102.

[2] F.B. Weissler, *Semilinear evolution equations in Banach spaces*, J. Funct. Anal. **32** (1979), 277–296.