Local and global existence for semilinear parabolic equations in terms of fractional powers of operators

Keisuke Nagaoka
(Tokyo University of Science)

Let A be the negative generator of a bounded analytic semigroup $\{e^{-tA}\}$ on a Banach space X and A^γ be its fractional power. Then we consider the abstract Cauchy problem:

$$(\mathbf{P}) \qquad \begin{cases} u'(t) + Au(t) = f(u(t)), \ t > 0, \\ u(0) = \phi, \end{cases}$$

where f is a nonlinear operator in X. Assume that the following conditions are satisfied:
($\mathbf{A}$) there exist constants $\lambda > 0$ and $M > 0$ such that $\|e^{-tA}v\| \le Me^{-\lambda t}\|v\| \ \forall\, v \in X, \forall\, t > 0$;
($\mathbf{F1}$) $f(0) = 0$;
($\mathbf{F2}$) there exist constants $C_0 > 0$, $\nu > 1$, and $0 \le \alpha < 1$ such that

$$\|f(v) - f(w)\| \le C_0(\|A^\alpha v\| + \|A^\alpha w\|)^{\nu-1}\|A^\alpha v - A^\alpha w\| \ \forall\, v, \forall\, w \in D(A^\alpha);$$

($\mathbf{\Phi}$) (**initial condition**) $\phi \in D(A^\theta)$ for $\theta \in [0, \alpha]$.

To solve ($\mathbf{P}$) we consider the solvability of the corresponding integral equation:

$$(\mathbf{IE}) \qquad u(t) = e^{-tA}\phi + \int_0^t e^{-(t-s)A} f(u(s))ds.$$

The purpose of this talk is to give a detailed proof of the local and global solvability of (**IE**) and (**P**). We slightly modify the presentation in [**1**].

Theorem. *Under the assumption stated above with* $1 - \alpha\nu + \theta(\nu - 1) \ge 0$ *and* $(\alpha - \theta)\nu < 1$ *one has the following assertions*:

1) (**local existence**) *There exists* $T > 0$ *such that* (**IE**) *has a local solution* $u \in C([0, T]; D(A^\theta))$ *satisfying* $\|A^\alpha u(t)\| \le C(\alpha)t^{-(\alpha-\theta)}e^{-\lambda t}\|A^\theta\phi\|$, $t \in (0, T]$, *with some constant* $C(\alpha) > 0$. *Moreover,* u *is a strong solution of* (**P**).
2) (**global existence**) *There exists* $\delta > 0$ *such that if* $\|A^\theta\phi\| \le \delta$, *then* (**IE**) *has a global solution* $u \in C([0, \infty); D(A^\theta))$ *satisfying* $\|A^\alpha u(t)\| \le C(\alpha)t^{-(\alpha-\theta)}e^{-\lambda t}\|A^\theta\phi\|$, $t \in [0, \infty)$, *with some constant* $C(\alpha) > 0$. *Moreover,* u *is a strong solution of* (**P**).
3) (**uniqueness**) *In the case where* $1 - \alpha\nu + \theta(\nu - 1) > 0$, *a solution of* (**IE**) *is unique in the class of* u *such that* $t^{\alpha-\theta}u \in BC((0, T]; D(A^\alpha))$. *In the case where* $1 - \alpha\nu + \theta(\nu - 1) = 0$, *it is unique in the class of* u *such that* $t^{\alpha-\theta}u \in BC((0, T]; D(A^\alpha))$ *and* $\lim_{t\to 0} t^{\alpha-\theta}\|A^\alpha u(t)\| = 0$.

References

[1] H. Hoshino, Y. Yamada, *Solvability and smoothing effect for semilinear parabolic equations*, Funkcial. Ekvac. **34** (1991), 475–494.