Local existence and uniqueness of solution to second sound equation in one space dimension

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We consider the following Cauchy problem (1) for the nonlinear wave equation $(1)_a$:

$$(1) \qquad \begin{cases} \partial_t^2 u = u\partial_x(u\partial_x u), \quad (t,x) \in (0,T] \times \mathbb{R}, & (1)_a \\ u(0,x) = \varphi(x), \quad x \in \mathbb{R}, & (1)_b \\ \partial_t u(0,x) = \psi(x), \quad x \in \mathbb{R}, & (1)_c \end{cases}$$

where $u(t,x)$ is unknown real valued function.
We call the equation $(1)_a$ second sound equation, which describes the wave of the temperature(entropy) in the superfluid.
We denote H^s as Sobolev space $(1-\partial_x^2)^{-s/2}L^2(\mathbb{R})$.

Theorem 1 (Keiichi Kato and S)**.** *Let $\varphi \in C^1 \cap L^\infty$ and $\partial_x\varphi$, $\psi \in H^s$ with $s > \frac{1}{2}$. Suppose that there exists a positive constant A such that $\varphi(x) \geq A > 0$ for $\forall x \in \mathbb{R}$. Then there exist $T > 0$ and a unique solution u of (1) such that*

$$u - \varphi \in \bigcap_{j=0,1,2} C^j([0,T]; H^{s-j+1}) \ \text{and}\ u(t,x) \geq A/2 \ \text{for}\ (t,x) \in [0,T] \times \mathbb{R},$$

where T depends only on $\|\varphi\|_{C^1}$, $\|\partial_x\varphi\|_{H^s}$, $\|\psi\|_{H^s}$and A.

Theorem 2 (Keiichi Kato and S)**.** *Suppose that $\varphi \in C^1 \cap L^\infty$, $\partial_x\varphi \in H^s$ and $\psi \in H^s$ with $s > \frac{1}{2}$, $\varphi(x) \geq A > 0$ for $\forall x \in \mathbb{R}$. For any number $T > 0$ and $A > 0$, there exists a number $\delta > 0$ such that if both $\|\psi\|_{L^2}$ and $\|\varphi(x)\partial_x\varphi(x)\|_{L^2}$ are less than δ, then the solution of (1) satisfying $u - \varphi \in \bigcap_{j=0,1,2} C^j([0,T]; H^{s-j+1})$ is unique.*

In [1], T. J. R. Hughes, T. Kato and J. E. Marsden prove the well-posedness of some class of second order quasi-linear hyperbolic equations including $(1)_a$. They obtain their result as an application of their abstract theorem. However, we can not apply their theorem to our problem, since the initial data φ is not L^2 integrable. We can apply the method of [1] to our existence problem for sufficiently smooth initial data.
In order to prove the uniqueness, one assume the restriction on the size of the solutions in [1]. On the other hand, by a priori estimate, we give the result of uniqueness under the restriction on the size of initial data, instead of the solutions.
In [2], P. Zhang and Y. Zheng treat the nonlinear wave equation $\partial_t^2 u = c(u)\partial_x(c(u)\partial_x u)$ on the condition that $0 <^\exists C_1 \leq C(u) \leq^\exists C_2$ for $u \in \mathbb{R}$, which does not include $(1)_a$.

References

[1] T. J. R. Hughes, T. Kato, J. E. Marsden, *Well-posed Quasi-linear Second-order Hyperbolic Systems with Applications to Nonlinear Elastodynamics and General Relativity*, Arch Rat. Mech. Anal 63. (1976), 273–294.

[2] P. Zhang, Y. Zheng, *Rarefactive solutions to a nonlinear variational wave equation of liquid crystals*, Comm. Partial differential equations 26. (2001), 381–419.