Theoretical and numerical results for standing waves of the nonlinear Schrdinger equation with harmonic potential and Sobolev subcritical/supercritical nonlinearities.

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In this talk, we deal with radially symmetric standing waves for the nonlinear Schrdinger equation with harmonic potential and critical/subcritical pure power nonlinearity.:

$$i\frac{\partial \psi}{\partial t} + \Delta\psi - \|x\|^2\psi + |\psi|^{2\sigma}\psi = 0,\ \ \psi = \psi(t,x), \quad \sigma \leq 2/(d-2), \quad x \in \mathbb{R}^d, \quad t > 0. \tag{1}$$

This equation arises in a wide variety of applications and is known as the Gross-Pitaevskii equation in the context of Bose-Einstein condensates with parabolic traps.

In the sub-critical case, Kavian and Weissler (Michigan Math. J. 1994) have proved the existence of infinitely many stationary localized solutions of (1) by variational methods. But this variational characterization does not give detailed information about the shape of the solution, and in particular it was an open question to know whether radial localized states exist with prescribed numbers of zeros, contrarily to the canonical (NLS) equation. So we study the existence of these nodal stationary waves for equation (1) by an alternative characterization of variational solutions using the bifurcation theory. Thereby, both global and local bifurcation behaviors were determined showing the existence of infinitely symmetric localized states (see [2]).

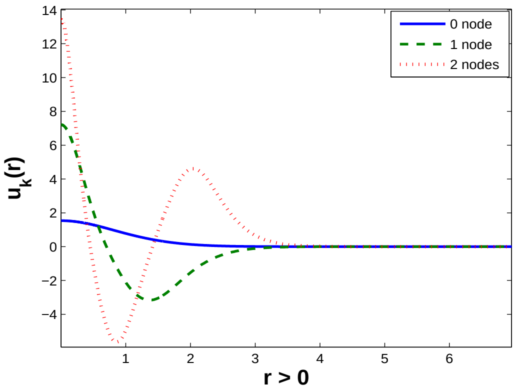

Figure 1: Plots of the first three stationary states of (1) ($d = 3$, $\sigma = 0.75$, $\omega = -1$).

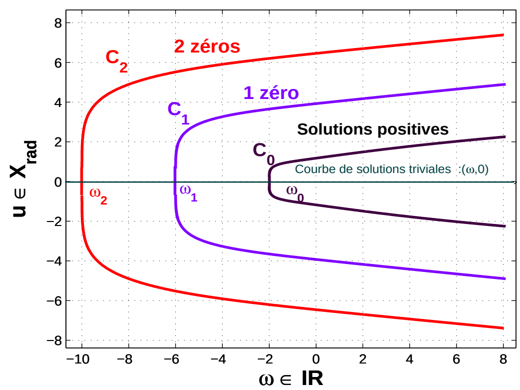

Figure 2: Bifurcating branches C_0, C_1, and C_2 : $d = 2$, $\sigma = 1.5 < \sigma^*$.

In the supercritical case (when $\sigma > 2/(d-2)$), since the Sobolev embedding does not hold, we cannot use the variational method directly. Nevertheless, it is already shown by Rabinowitz [J. Funct. Anal. 1971] the existence of global bifurcation solution branch for a semilinear elliptic equation on a bounded domain for any large nonlinear exponent σ. However it is essentially used the fact that the Haolder continuous function space is continuously embedded in the $L2$ space if the domain is bounded. Namely, the boundedness of the domain is crucial for their proof. Thus, it seems that this method does not work for the equation (1). To overcome this difficulty, we considered a modified equation obtained using a smooth function with compact support. Then applying the local bifurcation theorem by Crandall and Rabinowitz to the obtained equation, we have seen that there exists a local bifurcation branch for (1) with the modified nonlinearity. Later, by the Moser's iteration argument, we proved that this curve is a local bifurcation branch for the original equation. Finally, we have shown the existence of the global bifurcation branch by contradiction using a continuation argument (see Figures 3 and 4). These results were obtained in collaboration with Hiroaki Kikuchi (see [1]).

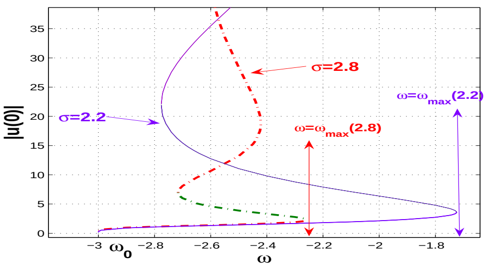

Figure 3: Bifurcating branches in the supercritical case : $d = 3$, $\sigma = 2.2; 2.8$.

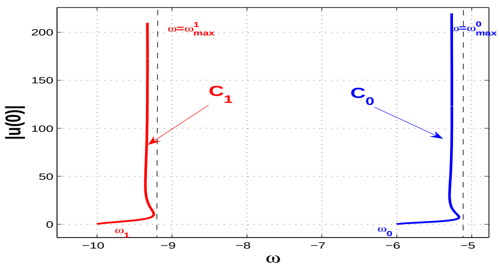

Figure 4: Bifurcating branches in the supercritical case : $d = 6$, $\sigma = 1$.

After presenting our theoretical results, numerical computations are finally presented in order to provide an illustration of the theoretical results that have been obtained and also to investigate several open questions for which only few results are known. The main step is to prove that the shooting algorithm enables to compute localized solutions with a prescribed number of zeros for suitable values of frequencies and any values of $\sigma > 0$ (see [3]).

Références

[1] Hadjselem F., Kikuchi H., *Existence and non-existence of solution for semilinear elliptic equation with harmonic potential and Sobolev critical/supercritical nonlinearities*, Journal of Mathematical Analysis and Applications, 746-754, 2011.

[2] Hadjselem F., *Radial solutions with prescribed numbers of zeros for the nonlinear Schrödinger equation with harmonic potential*, Nonlinearity 24, 1795-1819, 2011.

[3] Hadjselem F., Di Menza L., *Numerical study of solitons for nonlinear Schrödinger Equation with harmonic potential*, preprint.

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