On the global existence of spatially periodic solutions to a class of complex Ginzburg-Landau equations

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In this talk we consider the Cauchy problem for a class of complex Ginzburg-Landau equations

$$\text{(CGL)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = (\delta_1 + i\delta_2)\Delta u - i\mu|u|^{2\sigma}u, & (t,x) \in (0,\infty) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

where $i = \sqrt{-1}$, $\sigma > 0$, $\delta_1 > 0$, $\delta_2, \mu \in \mathbb{R}$ and $d \in \mathbb{N}$. We discuss the existence and uniqueness of global solutions to (CGL) with initial value $u_0 \in X_2^1(\mathbb{R}^d)$, where $X_p^m(\mathbb{R}^d)$ ($m \in \mathbb{N} \cup \{0\}$, $1 \le p \le \infty$) is the Sobolev space of spatially periodic functions defined as follows:

$$X_p^m(\mathbb{R}^d) := \left\{ u \in W_{\mathrm{loc}}^{m,p}(\mathbb{R}^d);\ u(\cdot) = u(\cdot + n) \text{ for all } n \in \mathbb{Z}^d \right\},$$

$$\|u\|_{m,p} := \Big(\sum_{|\alpha| \le m} \int_{(0,1)^d} |D^\alpha u(x)|^p dx \Big)^{1/p} \ (1 \le p < \infty),\ \ \|u\|_{m,\infty} := \max_{|\alpha| \le m} \Big(\operatorname*{ess.sup}_{x \in (0,1)^d} |D^\alpha u(x)| \Big).$$

Definition. A function u is said to be a *global solution* to (CGL) if (i) and (ii) are satisfied:

(i) $u \in C([0,\infty); X_2^1(\mathbb{R}^d)) \cap C((0,\infty); X_2^2(\mathbb{R}^d)) \cap C^1((0,\infty); X_2^0(\mathbb{R}^d))$;

(ii) u satisfies (CGL) on $(0,\infty)$ in $X_2^0(\mathbb{R}^d)$.

Gao and Wang [**1**] established the existence and uniqueness of global solutions to (CGL) in the d-dimensional torus $\mathbb{T}^d$. If we regard functions on $\mathbb{T}^d$ as periodic functions on $\mathbb{R}^d$, then their result is translated as follows:

Gao-Wang's result. Let $\delta_1 > 0$, $\delta_2 \in \mathbb{R}$ and $\sigma \in \mathbb{N}$. Assume that

$$(*) \qquad 2 - \frac{2}{\sqrt{1 + (\delta_2/\delta_1)^2} + 1} < p < 2 + \frac{2}{\sqrt{1 + (\delta_2/\delta_1)^2} - 1} \quad \text{and} \quad p > \sigma d.$$

Then for $u_0 \in X_p^1(\mathbb{R}^d)$ there exists a unique global solution to (CGL).

We focus our eyes on the case $p = 2$. In this case, σ and d satisfy $(*)$ only when $\sigma = d = 1$. Namely, Gao and Wang have not dealt with the case $d \ge 2$ or $\sigma \ne 1$.

The purpose of this talk is to relax the second condition in $(*)$ when $p = 2$ and to extend the restriction from $\sigma \in \mathbb{N}$ to $\sigma > 0$. Assuming further that $\delta_2 \mu > 0$, we can obtain the global existence and uniqueness of solutions to (CGL) even when $d \ge 2$ or $\sigma \ne 1$.

Main Theorem.

Let $\delta_1 > 0$, $\delta_2\mu > 0$ and

$$0 < \sigma < \infty \ (d = 1,2), \quad 0 < \sigma < \frac{1}{d-2} \ (d \ge 3).$$

Then for $u_0 \in X_2^1(\mathbb{R}^d)$ there exists a unique global solution to (CGL).

References

[1] H. Gao, X. Wang, *On the global existence and small dispersion limit for a class of complex Ginzburg-Landau equations*, Math. Methods Appl. Sci. **32** (2009), 1396–1414.