Asymptotic stability of stationary waves for symmetric hyperbolic-parabolic system in half space

Shinya Nishibata
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology

1 Introduction

We study large-time behavior of solutions to a system of viscous conservation laws over one-dimensional half space $\mathbb{R}_+ := (0, \infty)$,

$$f^0(u)_t + f(u)_x = (G(u)u_x)_x, \quad x \in \mathbb{R}_+, \ t > 0. \tag{1}$$

Here $u = u(t, x)$ is an unknown m-vector function taking values in an open convex set $\mathcal{O} \in \mathbb{R}^m$; $f^0(u)$ and $f(u)$ are smooth m-vector functions; $G(u)$ is a smooth $m \times m$ real matrix function. We assume that $\det D_u f^0(u) \neq 0$ holds for $u \in \mathcal{O}$, and that $G(u)$ is a non-negative matrix given by

$$G(u) = \begin{pmatrix} 0 & 0 \\ 0 & G_2(u) \end{pmatrix},$$

where $G_2(u)$ is an $m_2 \times m_2$ $(0 < m_2 < m)$ real matrix function and non-singular for $u \in \mathcal{O}$. Thus the system (1) consists of m_1-hyperbolic equations and m_2-parabolic equations where $m_1 := m - m_2$.

Assuming the system (1) has a strictly convex entropy. we deduce (1) to the symmetric system

$$A^0(u)u_t + A(u)u_x = B(u)u_{xx} + g(u, u_x), \tag{2}$$

where $A^0(u)$ is a real symmetric and positive matrix, $A(u)$ is a real symmetric matrix, $B(u)$ is a real symmetric and non-negative matrix, and $g(u, u_x)$ is non-linear terms. Moreover, under suitable conditions, the system (2) is rewritten to the decomposed form

$$A_1^0(u)v_t + A_{11}(u)v_x + A_{12}(u)w_x = g_1(u, w_x), \tag{3a}$$
$$A_2^0(u)w_t + A_{21}(u)v_x + A_{22}(u)w_x = B_2(u)w_{xx} + g_2(u, u_x), \tag{3b}$$

where v and w are unknown m_1- and m_2-vector functions respectively, given by $u = {}^T(v, w)$. In the system (3), $A_1^0(u)$ and $A_2^0(u)$ are real symmetric positive matrices; $A_{ij}(u)$ $(i, j = 1, 2)$ are real matrices satisfying

$$A(u) = \begin{pmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & A_{22}(u) \end{pmatrix},$$

and $A(u)$ is symmetric, i.e., $A_{11}(u)$ and $A_{22}(u)$ are symmetric and $A_{21}(u) = {}^TA_{12}(u)$; $B_2(u)$ is a real symmetric positive matrix; $g_1(u, w_x)$ and $g_2(u, u_x)$ are non-linear terms. We prescribe the initial and boundary conditions.

$$u(0, x) = u_0(x) = {}^T(v_0, w_0)(x), \tag{4}$$
$$w(t, 0) = w_{\rm b}, \tag{5}$$

where $w_{\mathrm{b}} \in \mathbb{R}^{m_2}$ is a constant. We assume that a spatial asymptotic state of the initial data is a constant:

$$\lim_{x\to\infty} u_0(x) = u_+ = {}^T(v_+, w_+), \quad \text{i.e.,} \quad \lim_{x\to\infty} (v_0, w_0)(x) = (v_+, w_+).$$

We suppose that

[**A**1] The matrices $A_{11}(u_+)$ is negative definite.

For the system (1) in the full space $\mathbb{R}^n$, Umeda–Kawashima–Shizuta [3] prove the asymptotic stability of a constant state under the stability condition. The main purpose of the present talk is to show the asymptotic stability of the stationary solution in the half space $\mathbb{R}_+$ under the stability condition.

2 Stationary solution

The stationary solution $\tilde{u}(x) = {}^T(\tilde{v}, \tilde{w})(x)$ is defined as a solution to (1) independent of t. Thus $\tilde{u} = {}^T(\tilde{v}, \tilde{w})$ satisfies equations

$$f(\tilde{u})_x = (G(\tilde{u})\tilde{u}_x)_x, \quad \text{i.e.,} \quad \begin{cases} f_1(\tilde{v}, \tilde{w})_x = 0, \\ f_2(\tilde{v}, \tilde{w})_x = (G_2(\tilde{u})\tilde{w}_x)_x, \end{cases} \tag{6}$$

where $f = {}^T(f_1, f_2)$. The boundary conditions are prescribed as

$$\tilde{w}(0) = w_{\mathrm{b}}, \quad \lim_{x\to\infty} \tilde{u}(x) = u_+. \tag{7}$$

Under the assumption

[**A**2] $\det D_v f_1(v_+, w_+) \neq 0$,

we solve the first equation in (6) by the implicit function theorem. Then there exists $V = V(\tilde{w})$ satisfying $f_1(V(\tilde{w}), \tilde{w}) = f_1(v_+, w_+)$. Let $\mu_j(w)$ $(j = 1, \ldots, m_2)$ be eigenvalues of the matrix $\tilde{A}(w) := G_2(u_+)^{-1} D_w f_2(V(w), w)$ and let $r_j(w)$ be corresponding eigenvectors. We solve the boundary value problem (6) and (7) under the following assumptions.

[**A**3] Eigenvalues of $\tilde{A}(w)$ are distinct, i.e., $\mu_1(w) > \mu_2(w) > \cdots > \mu_{m_2}(w)$.

[**A**4] $\mu_1(w_+) \leq 0$.

Theorem 1. *Assume that $\delta := |w_{\mathrm{b}} - w_+|$ is sufficiently small.*

(i) *For the case of $\mu_1(w_+) < 0$, there exists a unique smooth solution $\tilde{u}(x)$ to* (6) *and* (7) *satisfying*

$$|\partial_x^k(\tilde{u}(x) - u_+)| \leq Ce^{-cx} \quad \textit{for} \quad k = 0, 1, \ldots.$$

(i) *For the case of $\mu_1(w_+) = 0$, there exists a region $\mathcal{M} \subset \mathbb{R}^{m_2}$ such that if $w_{\mathrm{b}} \in \mathcal{M}$ and $D_w\mu_1(w_+) \cdot r_1(w_+) \neq 0$, then there exists a unique smooth solution $\tilde{u}(x)$ satisfying*

$$|\partial_x^k(\tilde{u}(x) - u_+)| \leq C\frac{\delta^{k+1}}{(1 + \delta x)^{k+1}} + Ce^{-cx} \quad \textit{for} \quad k = 0, 1, \ldots.$$

3 Stability of stationary solution

We show the asymptotic stability of non-degenerate stationary solution, of which existence is shown in Theorem 1-(i), under the stability condition [SK]:

[SK] Let $\lambda A^0(u_+)\phi = A(u_+)\phi$ and $B(u_+)\phi = 0$ for $\lambda \in \mathbb{R}$ and $\phi \in \mathbb{R}^m$. Then $\phi = 0$.

Notice that Shizuta–Kawashima [2] prove the equivalence of the condition [SK] and the following condition [K]:

[K] There exists an $m \times m$ real matrix K such that $KA^0(u_+)$ is skew-symmetric and $[KA(u_+)] + B(u_+)$ is positive definite, where $[A] := (A + {}^T\!A)/2$ is a symmetric part of a matrix A.

Theorem 2. *Let $\tilde{u}(x)$ be a non-degenerate stationary solution shown in Theorem* 1-(i)*. Assume that the condition* [SK] (*or* [K]) *holds. Then there exists a positive constant ε_1 such that if*

$$\|u_0 - \tilde{u}\|_{H^2} + \delta \le \varepsilon_1,$$

the problem (3), (4) *and* (5) *has a unique solution $u(t,x)$ globally in time satisfying*

$$u - \tilde{u} \in C([0,\infty), H^2(\mathbb{R}_+)).$$

Moreover the solution u converges to the stationary solution $\tilde{u}$:

$$\lim_{t\to\infty} \|u(t) - \tilde{u}\|_{L^\infty} = 0.$$

The crucial point of a proof of Theorem 2 is to obtain a uniform a priori estimate of a perturbation from the stationary solution. Let $(\varphi, \psi) := (v, w) - (\tilde{v}, \tilde{w})$ be a perturbation and $(\varphi_0, \psi_0) := (v_0, w_0) - (\tilde{v}, \tilde{w})$ be an initial perturbation. Under the assumption that $\sup_{0\le\tau\le t} \|(\varphi,\psi)(\tau)\|_{H^2} + \delta$ is sufficiently small, we obtain

$$\|(\varphi,\psi)(t)\|_{H^2}^2 + \int_0^t \left(\|\varphi_x(\tau)\|_{H^1}^2 + \|\psi_x(\tau)\|_{H^2}^2 \right) d\tau \le C\|(\varphi_0,\psi_0)\|_{H^2}^2. \tag{8}$$

To obtain (8), we first get the basic L^2 estimate of (φ, ψ) by using the energy form. To get the estimate for the derivatives of (φ, ψ), we use Matsumura–Nishida's energy method in half space developed in [1]. Finally, using the assumption [K], we obtain the dissipative estimate of φ_x. Combining these computations, we prove (8).

Acknowledgment. The present talk is based on the joint research with **Dr. Tohru Nakamura** at Kyushu University.

References

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