Existence of solutions to some degenerate parabolic equation associated with the p-Laplacian in the critical case

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Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}$). In this talk we consider the existence of solutions to the following initial-boundary value problem for a degenerate parabolic equation:

$$\text{(P)} \qquad \begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \Delta_p u(x,t) - |u|^{q-2}u(x,t) = f(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0(\cdot) \in L^r(\Omega), \end{cases}$$

where $T > 0$, $2 \le p, q, r < +\infty$, $p < q$, $f : \Omega \times (0,T) \to \mathbb{R}$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and

$$\text{(1)} \qquad r = N(q-p)/p.$$

When $p = 2$ and $f \equiv 0$, Weissler [3] and Brézis-Cazenave [2] obtained the following results: if $r \ge N(q-2)/2$, then (P) admits a unique local (in time) solution for any $u_0 \in L^r(\Omega)$; otherwise, (P) has no local solution in any usual sense for some initial data.

On the other hand, when $p \ge 2$, Akagi [1] established the following existence result: if $r > N(q-p)/p$, then (P) admits a local (in time) solution without any smallness on u_0 and f. However, no one has given an answer to the following natural question:

Open problem. *Does* (P) *admit a solution even in the critical case* $r = N(q-p)/p$?

Namely, the local (in time) existence in the critical case (1) is still left as an open problem. The purpose of this talk is to solve this open problem under a certain restriction on u_0.

To state our main result, we denote by $p' := p/(p-1)$ the Hölder conjugate of p.

Main Theorem.

Let $p, q, r \in [2, +\infty)$ satisfy $p < q$ and (1). Suppose the following conditions are fulfilled:
(A1) $u_0 \in L^r(\Omega)$ and $\|u_0\|_{L^r}$ is sufficiently small;
(A2) $f \in W^{1,p'}(0,T; W^{-1,p'}(\Omega) + L^{r'}(\Omega)) \cap L^1(0,T; L^r(\Omega))$.
Then there exists a time $T_0 = T_0(\|u_0\|_{L^r}, f) > 0$ such that (P) admits at least one function $u \in C_w([0,T_0]; L^r(\Omega)) \cap C([0,T_0]; L^2(\Omega)) \cap L^p(0,T_0; W_0^{1,p}(\Omega)) \cap L^q(0,T_0; L^q(\Omega))$ satisfying $du/dt \in L^{q'}(0,T_0; W^{-1,p'}(\Omega) + L^{r'}(\Omega))$,

$$\int_\Omega \frac{du}{dt}(t) v \, dx + \int_\Omega \left|\nabla u(t)\right|^{p-2} \nabla u(t) \cdot \nabla v \, dx - \int_\Omega \left|u(t)\right|^{q-2} u(t) v \, dx = \int_\Omega f(t) v \, dx$$

for a.a. $t \in (0,T_0)$ and for all $v \in W_0^{1,p}(\Omega) \cap L^r(\Omega)$, and $u(t) \to u_0$ in $L^r(\Omega)$ as $t \to +0$.

References

[1] G. Akagi, *Local existence of solutions to some degenerate parabolic equation associated with the p-Laplacian*, J. Differential Equations **241** (2007), 359–385.

[2] H. Brézis, T. Cazenave, *A nonlinear heat equation with singular initial data*, J. Anal. Math. **68** (1996), 277–304.

[3] F. B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in L^p*, Indiana Univ. Math. J. **29** (1980), 79–102.