Finite time degeneracy of 1D quasilinear wave equations

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In this talk, we consider the Cauchy problem of the following quasilinear wave equation:

$$\begin{cases} \partial_t^2 u = \partial_x((1+u)^{2a}\partial_x u), \quad (t,x) \in (0,T] \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \\ \partial_t u(0,x) = u_1(x), \quad x \in \mathbb{R}, \end{cases} \tag{1}$$

where $u(t,x)$ is an unknown real valued function and $a > 0$. The equation in (1) has richly physical backgrounds. (e.g. Ames and Lohner [1] and Zabusky [5]).

Throughout this talk, we always assume that there exists a constant $c_0 > 0$ such that

$$1 + u_0(x) \geq c_0 \tag{2}$$

for all $x \in \mathbb{R}$. This assumption enables us to regard the equation in (1) as a strictly hyperbolic equation near $t = 0$. By the standard local existence theorem for strictly hyperbolic equations, the local solution of (1) with smooth initial data uniquely exists until the one of the following two phenomena occurs. The first one is the blow-up:

$$\overline{\lim_{t \nearrow T^*}} (\|\partial_t u(t)\|_{L^\infty} + \|\partial_x u(t)\|_{L^\infty}) = \infty.$$

The second is the degeneracy of the equation:

$$\lim_{t \nearrow T^*} \inf_{(s,x)\in[0,t]\times\mathbb{R}} 1 + u(s,x) = 0.$$

When the equation degenerates, the standard local existence theorem does not work since the equation in (1) loses the strictly hyperbolicity. The aim of this talk is to obtain a sufficient condition for the occurrence of the degeneracy of the equation in finite time.

Theorem 1. Let $(u_0, u_1) \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$. Suppose that the initial data (u_0, u_1) satisfy that (2),

$$u_1(x) \pm (1 + u_0(x))^a \partial_x u_0(x) \leq 0 \quad \text{for all} \quad x \in \mathbb{R} \tag{3}$$

and

$$\int_{\mathbb{R}} u_1(x)dx < \frac{-2}{a+1}. \tag{4}$$

Then there exists $T^* > 0$ such that a local unique solution $u \in C([0,T^*); H^3(\mathbb{R})) \cap C^1([0,T^*); H^2(\mathbb{R}))$ of (1) exists and

$$\lim_{t \nearrow T^*} 1 + u(t,x_0) = 0 \quad \text{for some } x_0 \in \mathbb{R}. \tag{5}$$

Remark 2. In Theorem 1, we do not assume $u_1 \notin L^1(\mathbb{R})$. (3) implies that u_1 is a non-positive function, from which, the left hand side of (4) is going to $-\infty$, if $u_1 \notin L^1(\mathbb{R})$. Hence the assumption (4) is always satisfied, if $u_1 \notin L^1(\mathbb{R})$.

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The equation in (1) is formally equivalent to the following 2×2 conservation system:

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} - \partial_x \begin{pmatrix} V \\ \dfrac{(1+U)^{2a+1}}{2a+1} \end{pmatrix} = 0$$

where $U(t,x) = u(t,x)$, $V(t,x) = \int_{-\infty}^{x} \partial_t u(t,y)dy$ and u is a solution to the equation in (1). The global existence of solution to more general 2×2 conservation systems has been known (e.g. Johnson [2] and Yamaguchi and Nishida [4]). By applying the global existence theorem, we can show the following proposition.

Proposition 3. Let $(u_0, u_1) \in H^3(\mathbb{R}) \times H^2(\mathbb{R})$. Suppose that the initial data (u_0, u_1) satisfy that (2), (3) and

$$\int_{\mathbb{R}} u_1(x)dx > \frac{-2}{a+1}.$$

Then (1) has a global unique solution $u \in C([0,\infty); H^3(\mathbb{R})) \cap C^1([0,\infty); H^2(\mathbb{R}))$ satisfying that there exists a constant $c_1 > 0$ such that

$$1 + u(t,x) \geq c_1$$

for all $(t,x) \in [0,\infty) \times \mathbb{R}$.

This proposition and our main theorem say that $\dfrac{-2}{a+1}$ is a threshold of $\displaystyle\int_{\mathbb{R}} u_1(x)dx$ separating the global existence of solutions (such that the equation does not degenerate) and the degeneracy of the equation under the assumption of (3). The assumption (3) means that Riemann invariants are decreasing functions with x at $t = 0$. If (3) is not satisfied, solutions can blow up in finite time.

We give a known result on the degeneracy of the following equation:

$$\partial_t^2 u = (1+u)^{2a} \partial_x^2 u + a\lambda(1+u)^{2a-1}(\partial_x u)^2, \tag{6}$$

where $0 \leq \lambda \leq 2$. When $\lambda = 0, 1$ or 2, the equation of (6) has some physical backgrounds. (6) with $\lambda = 2$ is the equation in (1). If $\lambda \neq 2$, then the equation of (6) does not have the structure of conservation system. In [3], the author has shown that the equation of (6) with $0 \leq \lambda < 2$ can degenerate regardless of $\int_{\mathbb{R}} u_1(x)dx$.

References

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