PHASE FIELD METHOD FOR VOLUME PRESERVING MEAN CURVATURE FLOW

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Let $U_t \subset \mathbb{R}^n$ be a bounded open set and have a smooth boundary M_t for $t \in [0,T)$. The family of hypersurfaces $\{M_t\}_{t\in[0,T)}$ in $\mathbb{R}^n$ is called the volume preserving mean curvature flow if the velocity v of M_t is given by

$$v = h - \langle h \rangle \quad \text{on } M_t, \tag{1}$$

where h is the mean curvature of M_t and $\langle h \rangle$ is the mean value of h, that is,

$$\langle h \rangle := \frac{1}{\mathcal{H}^{n-1}(M_t)} \int_{M_t} h \, d\mathcal{H}^{n-1}.$$

Here $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. By (1), we have

$$\frac{d}{dt}\mathcal{L}^n(U_t) = \int_{M_t} v \, d\mathcal{H}^{n-1} = 0 \quad \text{(volume preserving property)}. \tag{2}$$

Next we mention the approximation of the volume preserving mean curvature flow via the phase field method. Let $\varepsilon \in (0,1)$ and $\Omega = \mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$. Rubinstein and Sternberg [4] considered the following Allen-Cahn equation with non-local term:

$$\begin{cases} \varphi^\varepsilon_t = \Delta\varphi^\varepsilon - \dfrac{W(\varphi^\varepsilon)}{\varepsilon^2} + \lambda^\varepsilon_{rs}, & (x,t) \in \Omega \times (0,\infty), \\ \varphi^\varepsilon(x,0) = \varphi^\varepsilon_0(x), & x \in \Omega, \end{cases} \tag{3}$$

where $W(s) = (1-s^2)^2/2$ and $\lambda^\varepsilon_{rs} = \lambda^\varepsilon_{rs}(t) := |\Omega|^{-1} \int_\Omega W'(\varphi^\varepsilon)/\varepsilon^2 \, dx$. Note that (3) has the volume preserving property, that is

$$\frac{d}{dt}\int_\Omega \varphi^\varepsilon \, dx = \int_\Omega \varphi^\varepsilon_t \, dx = 0. \tag{4}$$

But it is difficult to prove the existence of the global solution of the volume preserving mean curvature flow via (3) due to the Lagrange multiplier λ^ε_{rs}.

In this talk, we consider the following Allen-Cahn equation with non-local term:

$$\begin{cases} \varphi^\varepsilon_t = \Delta\varphi^\varepsilon - \dfrac{W(\varphi^\varepsilon)}{\varepsilon^2} + \lambda^\varepsilon \dfrac{\sqrt{2W(\varphi^\varepsilon)}}{\varepsilon}, & (x,t) \in \Omega \times (0,\infty), \\ \varphi^\varepsilon(x,0) = \varphi^\varepsilon_0(x), & x \in \Omega, \end{cases} \tag{5}$$

where

$$\lambda^\varepsilon = \lambda^\varepsilon(t) = \frac{-\int_\Omega \sqrt{2W(\varphi^\varepsilon)}\left(\Delta\varphi^\varepsilon - \frac{W'(\varphi^\varepsilon)}{\varepsilon}\right) dx}{2\int_\Omega \frac{W(\varphi^\varepsilon)}{\varepsilon} \, dx} = \frac{-2\int_\Omega \varphi^\varepsilon \left(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\right) dx}{\int_\Omega W(\varphi^\varepsilon)\, dx}. \tag{6}$$

Golovaty [2] studied the singular limit of the radially symmetric solutions for (5). Define $h(s) = s - \frac{1}{3}s^3$. Note that $h'(s) = \sqrt{2W(s)}$. (5) has the property similar to (4), that is,

$$\frac{d}{dt}\int_\Omega h(\varphi^\varepsilon)\, dx = \int_\Omega \sqrt{2W(\varphi^\varepsilon)}\varphi^\varepsilon_t \, dx = 0. \tag{7}$$

The following definition is similar to Brakke's solution [1]:

Definition 1 (L^2-flow [3])**.** Let $\{\mu_t\}_{t\in(0,T)}$ be a family of Radon measures on $\mathbb{R}^n$. Set $d\mu = d\mu_t dt$. We call $\{\mu_t\}_{t\in(0,T)}$ L^2-flow if the following hold:

(1) μ_t is $(n-1)$-rectifiable and has a generalized mean curvature vector $H \in L^2(\mu_t)$ a.e. $t \in (0,T)$,

(2) and there exists $C>0$ and a vector field $V \in L^2(\mu, \mathbb{R}^n)$ such that

$$V(x,t) \perp T_x\mu_t \quad \text{for } \mu\text{-a.e. } (x,t) \in \mathbb{R}^n \times (0,T) \tag{8}$$

and

$$\Big| \int_0^T \int_\Omega (\eta_t + \nabla\eta \cdot V)\, d\mu_t dt \Big| \leq C\|\eta\|_{C^0(\mathbb{R}^n\times(0,T))} \tag{9}$$

for any $\eta \in C^1_c(\mathbb{R}^n \times (0,T))$. Here $T_x\mu_t$ is the approximate tangent plane of μ_t at x. Moreover $V \in L^2(\mu,\mathbb{R}^n)$ with (8) and (9) is called a generalized velocity vector.

Definition 2. Let φ^ε be a solution for (5). We define a Radon measure μ^ε_t and μ^ε by

$$\mu^\varepsilon_t(\phi) := \frac{1}{\sigma}\int_{\mathbb{R}^n} \phi\Big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big)dx, \quad \mu^\varepsilon(\psi) := \frac{1}{\sigma}\int_0^\infty \int_{\mathbb{R}^n} \psi\Big(\frac{\varepsilon|\nabla\varphi^\varepsilon|^2}{2} + \frac{W(\varphi^\varepsilon)}{\varepsilon}\Big)dxdt, \tag{10}$$

for any $\phi \in C_c(\mathbb{R}^n)$ and $\psi \in C_c(\mathbb{R}^n \times [0,\infty))$. Here $\sigma = \int_{-1}^1 \sqrt{2W(s)}\,ds$.

Theorem 3. Let $n = 2,3$ and $U_0 \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary M_0. Then the following hold:

(1) There exist a family of solutions $\{\varphi^{\varepsilon_i}\}_{i=1}^\infty$ for (5) and a family of Radon measures $\{\mu_t\}_{t\in[0,\infty)}$ on $\mathbb{R}^n$ such that
  (a) $\mu^\varepsilon_0 \to \mathcal{H}^{n-1}\lfloor_{M_0}$ as Radon measures.
  (b) $\mu^\varepsilon_t \to \mu_t$ as Radon measures for a.e. $t \in [0,\infty)$.
  (c) $\mu^\varepsilon \to \mu$ as Radon measures on $\Omega \times [0,\infty)$, where $d\mu = d\mu_t dt$.

(2) There exist $\lambda \in L^2_{loc}(0,\infty)$ and $\nu \in L^\infty(\mu)$ such that
  (a) $\sup_{\varepsilon\in(0,1)} \|\lambda^\varepsilon\|_{L^2(0,T)} < \infty$ and $\lambda^\varepsilon \rightharpoonup \lambda$ in $L^2(0,T)$ for any $T>0$.
  (b)
$$\lim_{\varepsilon\to 0} \int_{\Omega\times(0,T)} \Phi\cdot\nu^\varepsilon\, d\mu^\varepsilon = \int_{\Omega\times(0,T)} \Phi\cdot\nu\, d\mu \tag{11}$$
  for any $\Phi \in C_c(\mathbb{R}^n \times (0,T);\mathbb{R}^n)$ for any $T>0$.
  (c) $\{\mu_t\}_{t\in(0,\infty)}$ is a L^2-flow with a generalized velocity vector $V = H - \lambda\nu$.
  (d)
$$\int_{t_1}^{t_2} \int_\Omega V\cdot\nu\, d\|\nabla\varphi(\cdot,t)\|dt = 0 \tag{12}$$
  for any $0 \leq t_1 < t_2 < \infty$.

References

[1] Brakke, K. A., *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, N.J., (1978).

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