Nonexistence of local solutions for the Cauchy problem of semirelativistic equations

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In this talk, we consider the Cauchy problem for the semirelativistic equations

$$\begin{cases} i\partial_t u \pm (m^2 - \Delta)^{1/2} u = \lambda |u|^p, & t \in \mathbb{R},\ x \in \mathbb{R}, \\ u(0) = u_0, & x \in \mathbb{R}, \end{cases} \tag{1}$$

with $\lambda \in \mathbb{C}\backslash\{0\}$ and $m \in \mathbb{R}$, where $\partial_t = \partial/\partial t$ and $\Delta = \partial_x^2$ is the Laplacian in $\mathbb{R}$. Here $(m^2 - \Delta)^{1/2}$ is realized as a Fourier multiplier with symbol $\sqrt{m^2+\xi^2}$: $(m^2-\Delta)^{1/2} = \mathfrak{F}^{-1}\sqrt{m^2+\xi^2}\mathfrak{F}$, where $\mathfrak{F}$ is the Fourier transform defined by

$$(\mathfrak{F}u)(\xi) = \hat{u}(\xi) = (2\pi)^{-1/2} \int \exp(-ix\xi) u(x) dx.$$

We remark that the Cauchy problem such as (1) arises in various physical settings and accordingly, especially in the massless case, semirelativistic equations are also called half-wave equations, fractional Schrödinger equations, and so on, see [6, 12] and reference therein. Moreover, the semirelativistic equation with Hartree type nonlinearity is used as a model of Boson star. For related subjects, we refer the reader to [1, 2, 4] and reference therein.

Here, we are interested in the local solvability of the Cauchy problem of (1). In general spacial dimension d, by the standard contraction argument, for $s > d/2$ and $u_0 \in H^s(\mathbb{R}^d)$, we have the unique local solution to (1), where H^s is the usual inhomogeneous Sobolev space defined by $H^s = (1-\Delta)^{-s/2} L^2$. Moreover, (1) with $m = 0$ is invariant under the scale transformation

$$u_\sigma(t,x) = \sigma^{1/(p-1)} u(\sigma t, \sigma x)$$

with $\sigma > 0$. Then

$$\|u_\sigma(0)\|_{\dot{H}^s(\mathbb{R}^d)} = \sigma^{1/(p-1)+s-d/2} \|u(0)\|_{\dot{H}^s(\mathbb{R}^d)},$$

where $\dot{H}^s$ is the usual homogeneous Sobolev space defined by $\dot{H}^s = (-\Delta)^{-s/2} L^2$ and with

$$s = s_p := d/2 - 1/(p-1) < d/2,$$

$\dot{H}^s$ norm of initial data is also invariant. s_p is called scale critical exponent and with fixed p, (1) is expected to have local solution for any $H^s(\mathbb{R}^d)$ initial data with $s > s_p$. However, in [4], it is shown that for $d = 1$ and $s < 1/2$, the solution map (1) is not C^2 in $H^s(\mathbb{R})$. This means that it is impossible to obtain local solution to (1) by an iteration argument. In addition, nonexistence results for local and global solutions have been obtained by a test function method. There is a large literature on test function method and we refer the reader

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to [3, 7–11]. In [5], nonexistence result for global solutions to (1) with $m = 0$ and $d = 1$ is obtained for $1 < p \leq 2$ by changing (1) to the corresponding wave equation and a test function method for the wave equation. Recently, Inui shows that the large data blow-up for $s \geq s_p$ and nonexistence of local solutions for $s < s_p$ to (1) with $d \geq 1$ and $m \in \mathbb{R}$ in [10]. In [10], he used the transformation of [5] and more appropriate test functions than those of [5]. We also remark that similar nonexistence results are obtained for the Cauchy problem of nonlinear Schrödinger equations

$$i\partial_t u + \Delta u = \lambda |u|^p$$

by Inui and Ikeda in [7, 8] and Ikeda and Wakasugi in [9]. In the present paper, we show the nonexistence of local solutions to (1) with initial data in $H^s(\mathbb{R})$ with $s \leq 1/2$ also by a test function method.

To state our main result, we introduce the definition of local weak solutions of (1) and review the transformation of semirelativistic equations of [5, 10]. For $T > 0$, we define function spaces X and X_T for $T > 0$ as follows:

$$\begin{aligned} X &= C([0, \infty); H^1(\mathbb{R}; \mathbb{R})) \cap C^1([0, \infty); L^2(\mathbb{R}; \mathbb{R})), \\ X_T &= \{\psi \in X; \operatorname{supp} \psi \subset [0, T) \times \mathbb{R}\}. \end{aligned}$$

Let $(\cdot \mid \cdot)$ be the usual L^2 scalar product defined by $(f \mid g) = \int f\overline{g}$. Then we define weak local solutions to (1).

Definition 1. *Let $T > 0$ and $u_0 \in L^1_{\mathrm{loc}}$. We say that u is a local weak solution to* (1)*, if u belongs to $L^1_{\mathrm{loc}}([0, T); L^p(\mathbb{R}))$ and the following identity*

$$\int_0^T \big(u(t)\big| i\partial_t \psi(t) \pm (m^2 - \Delta)^{1/2}\psi(t)\big)dt = i(u_0|\psi(0)) + \lambda \int_0^T \big(|u(t)|^p\big|\psi(t)\big)dt \qquad (2)$$

holds for any $\psi \in X_T$, where the double-sign corresponds to the sign of (1)*.*

Since test function method seems to rely on locality property of operators, a serious difficulty to apply a test function method to (1) arises when we try to handle the nonlocal operator $(m^2 - \Delta)^{1/2}$. To overcome this difficulty, we apply $-\mathrm{Im}\overline{\lambda}(i\partial_t \mp (m^2 - \Delta)^{1/2})$ to (1) to obtain

$$\Box \mathrm{Im}(\overline{\lambda}u) + m^2 \mathrm{Im}(\overline{\lambda}u) = \partial_t^2 \mathrm{Im}(\overline{\lambda}u) - \Delta \mathrm{Im}(\overline{\lambda}u) + m^2 \mathrm{Im}(\overline{\lambda}u) = -|\lambda|^2 \partial_t |u|^p. \qquad (3)$$

We remark that this transformation is used in [5, 10] and is just an inverse operation of decomposition of Klein-Gordon equations to obtain semirelativistic equations.

The corresponding local weak solutions to (3) are defined as follows:

Definition 2. *Let $T > 0$ and $u_0 \in L^1_{loc}$. We say that u is a local weak solution to* (3)*, if u belong to $L^1_{\mathrm{loc}}([0, T); L^p(\mathbb{R}))$ and the following identity*

$$\begin{aligned} &\int_0^T \big(\mathrm{Im}(\overline{\lambda}u)(t)\big|\Box\psi(t) + m^2\psi(t)\big)dt \\ &= \pm\big(\mathrm{Re}(\overline{\lambda}u_0)\big|(m^2 - \Delta)^{1/2}\psi(0)\big) + \big(\mathrm{Re}(i\overline{\lambda}u_0)\big|\partial_t\psi(0)\big) \\ &+ |\lambda|^2 \int_0^T \big(|u(t)|^p\big|\partial_t\psi(t)\big)dt \end{aligned} \qquad (4)$$

holds for $\psi \in C^2(\mathbb{R}^2;\mathbb{R})$ *with* $\operatorname{supp}\psi \subset [0,T)\times\mathbb{R}$*, where the double-sign corresponds to the sign of* (1)*.*

Since it is shown that local weak solutions to (1) are also those to (3) in [5,10], we consider the existence of local weak solutions to (3) and our main result is the following:

Theorem. *Let* $1 < p < \infty$ *and let* $f \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ *satisfy*

$$\exists\delta > 0 \text{ s.t. } f > 0 \text{ on } (-\delta,\delta) \text{ and } f \text{ is decreasing on } (0,\delta), \tag{5}$$

$$\lim_{\epsilon\searrow 0} f(\epsilon) = \infty. \tag{6}$$

Then there exists no $T > 0$ *such that there exists a local weak solution to* (1) *with* $u_0 = -i\overline{\lambda}^{-1}f$*.*

It is known that there exists $f \in H^{1/2}(\mathbb{R})$ such that f satisfies (5) and (6). Since $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ with $s > 1/2$, this means that $H^{1/2}(\mathbb{R})$ is the threshold of the local existence for the Cauchy problem of (1) if one tries to find local weak solutions to the initial data which belongs to a Sobolev space based on L^2.

At the end of this abstract, we give the main idea of the proof of the main theorem. To obtain the nonexistence results, we cancel the second derivatives of test functions and break the balance of scale for test functions by using a test function of the form

$$\psi(t,x) = \phi_1(t+x)\phi_2(t-x).$$

A direct calculation gives

$$\Box\psi(t,x) = 4\phi_1'(t+x)\phi_2'(t-x)$$

and ψ allows us to scale only ϕ_2 without any loss. We remark that with a test function of the form

$$\psi(t,x) = \phi_1(t)\phi_2(x),$$

scaling only ϕ_2 causes a loss and the nonexistence result by Inui in [10] seems to be optimal from the view point of the scale argument.

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