Non-uniqueness for elliptic operators with singular potentials[1]

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In this talk we discuss non-uniqueness for generators of positive C_0-semigroups on $L^p = L^p(\mathbb{R}^N)$ ($N \geq 1$, $1 < p < \infty$). We consider Schrödinger operators with inverse square potential

$$-Su(x) = \Delta u(x) - \frac{a}{|x|^2}u(x), \quad x \in \mathbb{R}^N \setminus \{0\},$$

where $a \in (-(\frac{N-2}{2})^2, -(\frac{N-2}{2})^2 + 1)$. Here we define the maximal operator $S_{p,\max}$ in L^p as

$$\begin{cases} S_{p,\max} := Su \quad \text{as a distribution in } \mathbb{R}^N \setminus \{0\}, \\ D(S_{p,\max}) := \{u \in L^p(\mathbb{R}^N) \ ; \ Su \in L^p(\mathbb{R}^N)\} \end{cases} \tag{0.1}$$

and the minimal operator $S_{p,\min}$ as

$$\begin{cases} S_{p,\min}u := Su, \\ D(S_{p,\min}) := C_0^\infty(\mathbb{R}^N \setminus \{0\}). \end{cases} \tag{0.2}$$

In [**2**], it is shown that $a \geq -(\frac{N-2}{2})^2$ is necessary and sufficient for the existence of positive distributional solutions of the corresponding parabolic problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \dfrac{a}{|x|^2}u = 0, \quad t \in \mathbb{R}^N \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \quad x \in \mathbb{R}^N. \end{cases} \tag{0.3}$$

We remark that if $a \geq -(\frac{N-2}{2})^2$, then we can construct the Friedrichs (selfadjoint) extension S_F of $S_{2,\min}$ via Hardy inequality:

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2}\,dx \leq \int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx, \quad u \in C_0^\infty(\mathbb{R}^N \setminus \{0\});$$

note that $S_F \subset S_{2,\max}$ and $-S_F$ generates a positive C_0-semigroup on L^2. Moreover, $S_{2,\min}$ is essentially selfadjoint if and only if $a \geq -(\frac{N-2}{2})^2+1$. In this case, the generator of positive C_0-semigroup on L^2 as an extension of $-S_{2,\min}$ is **unique** and $S_F = S_{2,\max}$.

Here we would like to mainly consider the **non-unique** case $a \in [-(\frac{N-2}{2})^2, -(\frac{N-2}{2})^2 + 1)$.

In [**1**], it is discussed in an abstract setting that if A (endowed with domain $D(A)$) is a generator of a C_0-semigroup on a Banach space X, then a restriction of A has a different extension B (endowed with the same domain $D(A)$) and B also generates

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[1] This talk is based on a joint work with Professors Giorgio Metafune and Chiara Spina (University of Salento).

a C_0-semigroup on X. However, the construction of another extension in [**1**] is not applicable to the operator having a differential expression.

The purpose of this talk is to give a technique for the construction of other extensions of $S_{2,\min}$ having the differential expression S. Moreover, we prove that the extensions generate positive C_0-semigroups on L^2.

The main theorem of this talk is the following:

Theorem 1.1 ([**4**]). *Let $a \in (-(\frac{N-2}{2})^2, -(\frac{N-2}{2})^2 + 1)$ and let S_F be the Friedrichs extension of $S_{2,\min}$. Set $\nu = \sqrt{a + (\frac{N-2}{2})^2}$ and $\varphi(x) = |x|^{-\frac{N-2}{2}} K_\nu(|x|)$ (K_ν: modified Bessel function of second kind). For $\alpha \in \mathbb{R}$, define*

$$\begin{cases} D(S_{2,\alpha}) := \left\{ \alpha \left(\int_{\mathbb{R}^N} \varphi(x) f(x)\, dx \right) \varphi + (1 + S_F)^{-1} f \in D(S_{2,\max}) \ ; \ f \in L^2 \right\}, \\ S_{2,\alpha} u = S_{2,\max} u. \end{cases}$$

Then $-S_{2,\alpha}$ generates a positive C_0-semigroup on L^2. Moreover, the spectrum of $S_{2,\alpha}$ is given by

$$\sigma(S_{2,\alpha}) = \begin{cases} [0,\infty) \cup \left\{ \left(1 - \frac{2\sin\nu\pi}{\alpha\pi}\right)^{1/\nu} \right\} & \text{if } \alpha \in (-\infty, 0) \cup \left(\frac{2\sin\nu\pi}{\pi}, \infty\right), \\ [0,\infty) & \text{if } \alpha \in \left[0, \frac{2\sin\nu\pi}{\pi}\right]. \end{cases}$$

The proof of Theorem 1.1 can be generalized to intermediate operators between a pair of operators $A_{\min}$ and $A_{\max}$ in a Banach space. Incidentally, the construction is also applicable to the one-dimensional Schrödinger operator with a potential given by Dirac measure δ (and $\beta \in \mathbb{R}$):

$$L_\beta = -\frac{d^2}{dx^2} + \beta\langle \cdot, \delta\rangle \delta \quad \text{in } \mathbb{R}$$

which is already considered in Kadowaki–Nakazawa–Watanabe [**3**].

We will also show the result (in L^p-case) for intermediate operators between $S_{p,\min}$ and $S_{p,\max}$ and for one-dimensional Schrödinger operators with a potential given by Dirac measure when $1 < p < \infty$.

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