Energy transfer model for the derivative nonlinear Schrödinger equations on the torus

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We consider the derivative nonlinear Schrödinger equations of the form on the torus:

$$\begin{cases} i\partial_t u + \partial_x^2 u = -i\lambda u^2 \partial_x \overline{u} + \mu |u|^4 u, & (t,x) \in \mathbb{R} \times \mathbb{T}, \\ u(0,x) = u_0(x), & x \in \mathbb{T}, \end{cases} \tag{1}$$

where $u = u(t,x) : [-T,T] \times \mathbb{T} \to \mathbb{C}$, $\lambda < 0$ (by rescaling), $\mu \in \mathbb{R}$ and $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the torus. The original derivative nonlinear Schrödinger equation in the completely integrable Hamiltonian systems takes forms as

$$i\partial_t u + \partial_x^2 u = i\lambda \partial_x(|u|^2 u), \tag{2}$$

and the deformed (equivalent) form, via a gauge transformation, of the equation (2) is

$$i\partial_t u + \partial_x^2 u = -i\lambda u^2 \partial_x \overline{u} - \frac{\lambda^2}{2}|u|^4 u - \frac{\lambda^2}{2\pi}\|u(t)\|_{L^2}^2 |u|^2 u - \frac{\lambda}{2\pi} u \int_0^{2\pi} \left(2\mathrm{Im}(u\partial_x \overline{u}) - \frac{\lambda}{2}|u|^4 \right) d\theta. \tag{3}$$

Each of the three equations (1), (2) and (3) possesses three conservation laws: L^2-norm, $\dot{H}^1$-energy, ($\dot{H}^{1/2}$-)momentum. In particular, the energy of (1) is defined by

$$E[u](t) = \int_0^{2\pi} \left(\frac{1}{2}|\partial_x u|^2 + \frac{\lambda}{4}|u|^2 \mathrm{Im}(\overline{u}\partial_x u) + \frac{\lambda^2 + 2\mu}{12}|u|^6 \right) dx,$$

which maintains positive for any $u \in H^1$, provided $\mu > -5\lambda^2/16$ (defocusing case); this is because the energy density is positive. One the other hand, the energies of (2) and (3) happen to be negative, that is possible if $u \in H^1$ with large L^2-norm. Exploiting this, one can establish the global in time well-posedness in the energy class H^1 for both type of equations (1) and (2) (e.g., [3]).

Let us concentrate on the equation (1) in the defocusing case, as a prototype model. We explore solutions exhibiting energy transfers (amplitude shift) among frequencies with a small cardinal number. It is worth noticing that the transfer of energy cascade to high frequencies was investigated in the cubic defocusing nonlinear Schrödinger equation on the two dimensional torus [1]. Recently, dynamical result similar to this was obtained in the quintic nonlinear Schrödinger equation on the one dimensional torus [2]. Such two equations, including the equation (1), come from the typical model classified as in the incompletely integrable Hamiltonian systems.

Definition 1 For $M, N \in \mathbb{Z}$ such that none of

$$M,\ N,\ M-N,\ 4M+N,\ M+4N,\ 2M+3N,\ 2N+3M,\ 7M+3N,\ 3M+7N,\ 6M-N,\ M-6N, \tag{4}$$

are zero and $M + N > 0$, we define resonant sets $\Lambda_{M,N}$ as follows:

$$\Lambda_{M,N} = \{\, M,\ N,\ -3M-N,\ -M-3N \,\}.$$

In particular, the case $M + N = 1$, $|M| \gg 1$ is acceptable in (4).

Our main theorem is as follows. The validity of our result seems to be weak by comparison with [1, 2] in some sense, due to the inevitable issues arising from loss of derivatives.

Theorem 1 *Assume $\mu/\lambda^2 \sim 1$ and $-5\lambda^2/16 < \mu \neq 0$. There exist $c_1 = c_1(M+N) > 0$ and $T(\sim 1/(M+N)^2)$-periodic function $K(t) : \mathbb{R} \to (0,1)$ satisfying $K(0) \le 1/2 - c_1$ and $K(T) \ge 1/2 + c_1$ so that there exists a solution to (1) which satisfies that for all $|t| \le T$ and $\delta \in (1/3, 1/2)$*

$$u(t,x) = \sqrt{\frac{-20(M+N)}{\lambda}} \left(\sum_{\xi \in \Lambda_{M,N}} a_\xi(t) e^{ix\xi} + e(t,x) \right),$$

where $c_1(M+N) \to 0$ *as* $M+N \to \infty$, *and*

$$|a_M(t)|^2 = \frac{|a_{-3M-N}(t)|^2}{2} = K(t), \quad |b_N(t)|^2 = \frac{|a_{-M-3N}(t)|^2}{2} = 1 - K(t),$$

$$e|_{t=0} = 0, \quad \sup_{|t|\le T} \sum_{\xi \in \Lambda^c_{M,N}} \langle \xi \rangle^\delta |\mathcal{F}_x e(t,\xi)| \lesssim \frac{1}{\min\{|M|,|N|\}^\delta}.$$

Roughly speaking, the proof consists of two parts. In the first part, we show the finite dimensional (order four) toy model dynamical system, which approximates the infinite dimensional system (1), that corresponds to resonant sets. In the second part, we construct error estimates on the non-resonant frequencies part that imply one can view the behavior of solution with the desired properties.

References

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