Symmetry-breaking for positive solutions of the one-dimensional Liouville type equation

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In this talk we consider the two-point boundary value problem for the one-dimensional Liouville type equation

$$u'' + \lambda|x|^l e^u = 0, \quad x \in (-1,1); \quad u(-1) = u(1) = 0, \tag{1}$$

where $\lambda > 0$ and $l > 0$. Jacobsen-Schmitt [1] presented the exact multiplicity result of radial solutions for the multi-dimensional problem

$$\Delta u + \lambda|x|^l e^u = 0 \ \text{ in } B; \quad u = 0 \ \text{ on } \partial B, \tag{2}$$

where $\lambda > 0$, $l \geq 0$ and $B := \{x \in \mathbf{R}^N : |x| < 1\}$, $N \geq 1$. In the case $N = 1$, problem (2) is reduced to (1). We note here that every solution of (2) is positive in B, by the strong maximum principle. Jacobsen and Schmitt [1] proved that if $1 \leq N \leq 2$, then there exists $\lambda_* > 0$ such that (2) has exactly two radial solutions for $0 < \lambda < \lambda_*$, a unique radial solution for $\lambda = \lambda_*$ and no radial solution for $\lambda > \lambda_*$. Recently, Miyamoto [3] proved the following symmetry-breaking bifurcation result for (2) when $N = 2$.

Theorem A ([3]). *Let n_0 be the largest integer that is smaller than $1 + \frac{l}{2}$ and let $\alpha_n := 2\log\frac{2l+4}{l+2-2n}$. All the radial solutions of (2) with $N = 2$ can be written explicitly as*

$$\lambda(\alpha) = 2(l+2)^2(e^{-\alpha/2} - e^{-\alpha}), \quad U(r;\alpha) = \alpha - 2\log(1 + (e^{\alpha/2} - 1)r^{l+2}).$$

The radial solutions can be parameterized by the L^∞-norm, it has one turning point at $\lambda = \lambda(\alpha_0) = (l+2)/2$, and it blows up as $\lambda \downarrow 0$. For each $n \in \{1, 2, \cdots, n_0\}$, $(\lambda(\alpha_n), U(r;\alpha_n))$ is a symmetry-breaking bifurcation point from which an unbounded branch consisting of non-radial solutions of (2) with $N = 2$ emanates, and $U(r;\alpha)$ is nondegenerate if $\alpha \neq \alpha_n$, $n = 0, 1, \cdots, n_0$. Each non-radial branch is in $(0, \lambda(\alpha_0))) \times \{u > 0\} \subset R \times H_0^2(B)$.

Korman [2] found the interesting property of radial solutions to (2). Let w be a unique solution of the initial value problem

$$w'' + |x|^l e^w = 0, \quad x > 0; \quad w(0) = w'(0) = 0.$$

It is easy to show that $w(x) < 0$, $w'(x) < 0$, $w''(x) < 0$ for $x > 0$ and $\lim_{x\to\infty} w(x) = -\infty$. Hence, there exists the inverse function η of $-w(x)$. It follows that $\eta \in C^2(0,\infty)$, $\eta(t) > 0$ and $\eta'(t) > 0$ for $t > 0$, $\eta(0) = 0$, and $\lim_{t\to\infty} \eta(t) = \infty$. By a direct calculation, we easily prove that, for each $\alpha > 0$,

$$U(x) = w(\eta(\alpha)|x|) + \alpha \tag{3}$$

satisfies $\|U\|_\infty = \alpha$ and is a positive even solution of (1) at $\lambda = [\eta(\alpha)]^{l+2}e^{-\alpha}$. By using this fact, we can show the following result, which is a generalization of the result by Jacobsen and Schmitt [1] for the case $N = 1$.

Proposition 1. *For each $\alpha > 0$, there exists a unique $(\lambda(\alpha), U(x;\alpha))$ such that (1) with $\lambda = \lambda(\alpha)$ has a unique positive even solution $U(x;\alpha)$ such that $||U||_\infty = \alpha$, $U(x;\alpha) \in C^2([-1,1] \times (0,\infty))$, $\lambda \in C^2(0,\infty)$, and*

$$\lim_{\alpha\to+0} \lambda(\alpha) = \lim_{\alpha\to\infty} \lambda(\alpha) = 0.$$

Moreover, there exist $\alpha_* > 0$ such that $\lambda'(\alpha) > 0$ for $0 < \alpha < \alpha_*$ and $\lambda'(\alpha) < 0$ for $\alpha > \alpha_*$. In particular, (2) has exactly two positive even solutions for $0 < \lambda < \lambda_*$, a unique positive even solution for $\lambda = \lambda_*$ and no positive even solution for $\lambda > \lambda_*$, where $\lambda_* = \lambda(\alpha_*)$.

Hereafter, let $\lambda(\alpha)$, $U(x;\alpha)$, α_* and λ_* be as in Proposition 1.

Let $m(\alpha)$ be the Morse index of $U(x;\alpha)$ to (1), that is, the number of negative eigenvalues μ of

$$\phi'' + \lambda|x|^l e^{U(x;\alpha)}\phi + \mu\phi = 0, \quad x \in (-1,1); \quad \phi(-1) = \phi(1) = 0. \tag{4}$$

A solution $U(x;\alpha)$ is said to be degenerate if $\mu = 0$ is an eigenvalue of (4). Otherwise, it is said to be nondegenerate.

The following result is the main result of this talk.

Theorem 1. *Let $(\lambda(\alpha), U(x;\alpha))$ and $\alpha_* > 0$ be as in Proposition 1. Then there exist constants α_1, α_2 and α_3 such that $\alpha_* < \alpha_1 \le \alpha_2 \le \alpha_3$ and the following* (i)–(vii) *hold*:

- (i) *if $0 < \alpha < \alpha_*$, then $m(\alpha) = 0$ and $U(x;\alpha)$ is nondegenerate*;
- (ii) *if $\alpha = \alpha_*$, then $m(\alpha) = 0$ and $U(x;\alpha)$ is degenerate*;
- (iii) *if $\alpha_* < \alpha < \alpha_1$, then $m(\alpha) = 1$ and $U(x;\alpha)$ is nondegenerate*;
- (iv) *if $\alpha = \alpha_1$, then $m(\alpha) = 1$ and $U(x;\alpha)$ is degenerate*;
- (v) *if $\alpha = \alpha_2$, then $m(\alpha) = 1$, $U(x;\alpha)$ is degenerate and $(\lambda(\alpha_2), U(x;\alpha_2))$ is a non-even bifurcation point, that is, for each $\varepsilon > 0$, there exists (u,λ) such that u is a positive non-even solution of* (1) *and $|\lambda - \lambda(\alpha_2)| + \|u - U(\,\cdot\,,\alpha_2)\|_\infty < \varepsilon$*;
- (vi) *if $\alpha = \alpha_3$, then $m(\alpha) = 1$ and $U(x;\alpha)$ is degenerate*;
- (vii) *if $\alpha > \alpha_3$, then $m(\alpha) = 2$ and $U(x;\alpha)$ is nondegenerate.*

Moreover, if $0 < \lambda < \lambda(\alpha_3)$, then (1) *has a positive non-even solutions $u(x)$ which satisfies $\lim_{\lambda\to+0} \|u\|_\infty = \infty$.*

Recalling the result by Jacobsen and Schmitt [1], the structures of radial solutions of (2) with $N = 2$ and even solutions of (1) seems to be same. However, in [3] Miyamoto proved the Morse index of the radial solution increases by one when α passes each α_n, $n = 0, 1, 2, \cdots, n_0$, where α_n is as in Theorem A. On the other hand, the Morse index of even solutions of (2) is at most 2 for each $l > 0$.

When $N = 2$, radial solutions of (2) can be written explicitly, and hence, Miyamoto [3] succeeded to find the bifurcation points. That is difficult even if we know exact solutions, much more difficult if we do not know it. When $N \ne 2$, we do not know exact radial solutions of (2) with $l > 0$. However, Korman [2] found the solution (3) as we have mentioned before. When $N = 1$, the structure of eigenvalues $\{\mu_k(\alpha)\}_{k=1}^\infty$ of (4) is well-known. Combining these facts, we can show (i)–(iv) of Theorem 1 and $m(\alpha) \le 2$ for $\alpha > 0$. Moreover, by using the comparison function introduced in [4], we can prove that $m(\alpha) \ge 2$ for all sufficiently large $\alpha > 0$. Then we can obtain a symmetry-breaking bifurcation point of (1), by using the Leray-Schauder degree, and hence we will obtain (v)–(vii) of Theorem 1.

References

[1] J. Jacobsen and K. Schmitt, The Liouville-Bratu-Gelfand problem for radial operators, *J. Differential Equations* **184** (2002), 283–298.

[2] P. Korman, Global solution curves for self-similar equations, *J. Differential Equations* **257** (2014) 2543- 2564.

[3] Y. Miyamoto, Nonradial maximizers for a Hénon type problem and symmetry breaking bifurcations for a Liouville-Gel'fand problem with a vanishing coefficient, *Math. Ann.* **361** (2015) 787–809.

[4] S. Tanaka, Morse index and symmetry-breaking for positive solutions of one-dimensional Hénon type equations, *J. Differential Equations* **255** (2013) 1709–1733.