PROF. YOSIDA'S PROOF OF THE PLANCHEREL AND THE BOCHNER THEOREMS FOR LOCALLY COMPACT ABELIAN GROUPS

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Banach algebras were introduced by M. Nagumo [6] and K. Yosida [8] in 1936. I. M. Gelfand's celebrated paper [1] appeared five years later. The theory was immediately applied to the reconstruction of the theory of locally compact abelian groups. In the Soviet Union, Gelfand, D. A. Raikov, M. Krein and M. Neumark published short notes in Doklady Nauk in 1940–1944 but the details were shown to the public only after the war as a book [3]. Under the isolation caused by the war, similar works were done in France by H. Cartan and R. Godement, and in the United States by I. E. Segal independently. They also had to wait until the end of the war before they published their results.

On the contrary, Japanese were able to read Doklady and publish their works. In 1944 Professor Yosida published two papers on this subject. In [10] he claimed that he improved Krein's proof [5] of the Plancherel theorem, and in [11] Raikov's proof [7] of the Bochner theorem. I think his proofs are the most natural ones but they did not attract due attention, probably because his proofs yet contained some gaps. I showed how to fill in the gaps in Comments of his Collected Papers [12] by quoting a book [4]. However, since I found a gap in the book again, I like to fill it in here.

Let X be a locally compact abelian group with a Haar measure dx. In the theory of Gelfand et al. they use the group ring $\mathbf{B}_0 = L^1(X)$ with convolution as multiplication. By adding the unity 1 if X is not discrete, we obtain a commutative Banach algebra $\mathbf{B}$ with 1. The starting point was Gelfand-Raikov's observation [2] that the space $\operatorname{Spec}\mathbf{B}$ of maximal ideals of $\mathbf{B}$ is identified with the character group $\hat{X}$ except for the point $M_\infty = L^1(X)$ at infinity in the indiscrete case. The Gelfand topology on $\hat{X}$ coincides with the compact open topology commonly used in the group theory. An explicit proof was not given but is not difficult.

The Banach algebra $\mathbf{B}$ has the involution defined by

$$f^*(x) = \overline{f(-x)}, \quad f \in L^1(X), \tag{1}$$

but it is not a C^*-algebra. This makes the theory difficult. Yosida's idea is to employ the C^*-algebra $\mathbf{A}$ generated by the convolution operators T_f on the Hilbert space $L^2(X)$ instead, where

$$T_f\phi(x) = f * \phi(x) = \int_X f(x-y)\phi(y)dy, \quad f \in L^1(X),\, \phi \in L^2(X). \tag{2}$$

Namely, let $\mathbf{A}_0$ be the closure of $\{T_f \mid f \in \mathbf{B}_0\}$ in the normed space $L(L^2(X))$ of all bounded linear operators, and consider the C^*-algebra $\mathbf{A}$ obtained by adding the identity I to $\mathbf{A}_0$ if X is not discrete. Yosida discussed the Fourier transform

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_X f(x)e^{-ix\xi}dx, \quad f \in L^1(X), \tag{3}$$

by factorizing it through the algebra $\mathbf{A}$ with the aid of the following.

Lemma 1. *The continuous homomorphism $T : \mathbf{B} \to \mathbf{A}$ defined by*

$$T(a1 + f) = aI + T_f \tag{4}$$

induces the homeomorphism

$$T^* : \operatorname{Spec}\mathbf{A} \cong \operatorname{Spec}\mathbf{B}. \tag{5}$$

Proof. Since the continuous linear operator T has a dense range, its dual $T' : \mathbf{A}' \to \mathbf{B}'$ is an injective continuous linear mapping in the weak topologies. Hence its restriction T^* to $\operatorname{Spec}\mathbf{A}$ is a homeomorphism onto the image. We have only to prove its surjectivity, and this part was missing in [10].

Let $\tau \in \hat{X}$. Then, the multiplication M_τ by the character $e^{ix\tau}$ defines the isomorphisms

$$M_\tau : L^1(X) \to L^1(X),$$
$$M_\tau : L^2(X) \to L^2(X).$$

If $f \in L^1(X)$, then we have

$$(T_{M_\tau f}\phi, \psi) = \int f(x-y)e^{i(x-y)\tau}\phi(y)\overline{\psi(x)}dydx = (M_\tau T_f M_\tau^{-1}\phi, \psi).$$

Taking the uniform limits, we have the isomorphism $\operatorname{Ad}_\tau : \mathbf{A} \to \mathbf{A}$ satisfying

$$\operatorname{Ad}_\tau A = M_\tau\, A\, M_\tau^{-1}, \quad A \in \mathbf{A}, \tag{6}$$

and the intertwining relation

$$T \circ M_\tau = \operatorname{Ad}_\tau \circ T. \tag{7}$$

Hence we obtain the commutative diagram of continuous mappings

$$\begin{array}{ccc} \operatorname{Spec}\mathbf{A} & \xrightarrow{T^*} & \operatorname{Spec}\mathbf{B} \\ \Big\downarrow \operatorname{Ad}_\tau^* & & \Big\downarrow M_\tau^* \\ \operatorname{Spec}\mathbf{A} & \xrightarrow{T^*} & \operatorname{Spec}\mathbf{B}. \end{array} \tag{8}$$

The homeomorphism T^* sends the point at infinity in $\operatorname{Spec}\mathbf{A}$ to the point at infinity in $\operatorname{Spec}\mathbf{B}$, and the nontrivial C^*-algebra $\mathbf{A}$ has a maximal ideal different from the point at infinity.

If $f \in L^1(X)$, then

$$\mathcal{F}(M_\tau f)(\xi) = \int f(x) e^{ix\tau} e^{-ix\xi} dx = \mathcal{F}f(\xi - \tau). \tag{9}$$

This shows that at a finite point $\xi \in \hat{X}$, we have $M_\tau^*(\xi) = \xi - \tau$. Since the action of $\hat{X}$ on $\hat{X}$ is transitive, the range of T^* coincides with $\operatorname{Spec}\mathbf{B}$.

Yosida had proved in [9] that the Gelfand representations

$$\mathbf{A} \cong C(\operatorname{Spec}\mathbf{A}), \tag{10}$$

$$\mathbf{A}_0 \cong C_\infty(\hat{X}) \tag{11}$$

are not only isomorphisms of C^*-algebras but also order preserving. That is, an $A \in \mathbf{A}$ is positive definite as an operator:

$$(A\phi, \phi) \geq 0, \quad \phi \in L^2(X), \tag{12}$$

if and only if its spectral representation is positive:

$$\hat{A}(\xi) \geq 0, \quad \xi \in \hat{X}. \tag{13}$$

Combining the $*$-isomorphism $T : L^1(X) \to \mathbf{A}_0$ of algebras with isomorphism (11), we obtain

Theorem 1 (Riemann-Lebesgue). *The Fourier transformation*

$$\mathcal{F} : L^1(X) \to C_\infty(\hat{X}) \tag{14}$$

is a continuous linear injection with dense range.

Remark that

$$(T_f\phi, \psi) = f * \phi * \psi^*(0), \quad f \in L^1(X),\ \phi, \psi \in L^2(X). \tag{15}$$

In particular, we have $\mathcal{F}f(\xi) \geq 0$ for any $\xi \in \hat{X}$ if and only if $f * \phi * \phi^*(0) \geq 0$ for any $\phi \in L^2(X)$.

In order to introduce a Haar measure $d\xi$ on $\hat{X}$, we consider the function space $\Lambda(X)$ of all linear combinations $\sum a_j \phi_j * \phi_j^*$ with $a_j \in \mathbf{C}, \phi_j \in L^1(X) \cap L^2(X)$. The space $\mathcal{K}(X)$ of all continuous functions with compact support in X is included in $L^1(X) \cap L^2(X)$ and is dense both in $L^1(X)$ and in $L^2(X)$. Hence it follows that $\Lambda(X)$, which is included in $L^1(X) \cap C_\infty(X) \subset L^2(X)$, is also dense both in $L^1(X)$ and in $L^2(X)$.

We denote its Fourier image by $\hat{\Lambda}(\hat{X})$:

$$\hat{\Lambda}(\hat{X}) = \mathcal{F}\Lambda(X).$$

Since $\Lambda(X)$ is dense in $L^1(X)$, it follows that $\hat{\Lambda}(\hat{X})$ is dense in $C_\infty(\hat{X})$.

Now, we define the linear functional J on $\hat{\Lambda}(\hat{X})$ by

$$J(\hat{f}) = f(0), \quad f \in \Lambda(X). \tag{16}$$

Krein [4] and Yosida [3] claimed that the functional J could be defined for a much wider function space but it is very doubtful.

The linear functional J is *positive* in the sense that

$$\hat{f}(\xi) \geq 0, \quad \xi \in \hat{X} \implies J(\hat{f}) \geq 0. \tag{17}$$

In fact, let $\delta_V \in \mathcal{K}(V)$ satisfy $\int \delta(x)dx = 1$. Then, as the neighborhood V of 0 shrinks to 0, we have, by (12) and (15),

$$J(\hat{f}) = \lim_{V \to 0} f * \delta_V * {\delta_V}^*(0) \geq 0.$$

Moreover, we have for any $f \in \Lambda(X)$

$$J(\hat{S}_\tau \hat{f}) = J(\hat{f}), \quad \tau \in \hat{X}, \tag{18}$$

$$J(\hat{M}_t \hat{f}) = f(t), \quad t \in X, \tag{19}$$

where $\hat{S}_\tau$ denotes the shift $(\hat{S}_\tau \hat{f})(\xi) = \hat{f}(\xi - \tau)$, and $\hat{M}_t$ the multiplication $\hat{M}_t \hat{f}(\xi) = e^{it\xi}\hat{f}(\xi)$. Equation (18) follows from (9), and Equation (19) from

$$(S_{-t}f)\hat{}(\xi) = e^{it\xi}\hat{f}(\xi). \tag{20}$$

If ϕ and ψ are in $L^1(X) \cap L^2(X)$, then the convolution $\phi * \psi^*$ belongs to $\Lambda(X)$ because we have

$$\begin{aligned} \phi * \psi^* = \frac{1}{4}\{&(\phi + \psi) * (\phi + \psi)^* - (\phi - \psi) * (\phi - \psi)^* \\ &+ i(\phi + i\psi) * (\phi + i\psi)^* - i(\phi - i\psi) * (\phi - i\psi)^*\}. \end{aligned} \tag{21}$$

In particular, we have $(a1 + f) * \phi * \phi^* \in \Lambda(X)$ for any $a \in \mathbf{C}, f \in L^1(X)$, and $\phi \in L^1(X) \cap L^2(X)$. The value of J at its Fourier transform $(a + \hat{f})|\hat{\phi}|^2$ is written

$$J((a + \hat{f})|\hat{\phi}|^2) = (a1 + f) * \phi * \phi^*(0) = ((aI + T_f)\phi, \phi).$$

Hence we have the estimates

$$|J((a + \hat{f})|\hat{\phi}|^2)| \leq \|aI + T_f\| \|\phi\|^2 = \|a + \hat{f}(\xi)\|_{C(\operatorname{Spec} \mathbf{A})} \|\phi\|^2_{L^2(X)}. \tag{22}$$

This shows that, when a $\phi \in L^1(X) \cap L^2(X)$ is fixed, the linear functional $J((a+\hat{f})|\hat{\phi}|^2)$ in $a+\hat{f}(\xi) \in C(\operatorname{Spec} \mathbf{A})$ can uniquely be extended to a continuous linear functional $J(c(\xi)|\hat{\phi}(\xi)|^2)$ in $c(\xi) \in C(\operatorname{Spec} \mathbf{A})$.

In order to show that the extension is a positive linear functional depending only on the function $c(\xi)|\hat{\phi}(\xi)|^2)$, we first consider the case where $c(\xi) \in C(\operatorname{Spec} \mathbf{A})$ satisfies $0 \leq c(\xi) < 1$. The square root $s(\xi) = \sqrt{c(\xi)}$ can be approximated uniformly by the Fourier image $\mathcal{F}\mathbf{B}$. If $|s(\xi) - (a_n + \hat{f}_n(\xi))| \leq 2^{-n-1}$, then we have $|c(\xi) - |a_n + \hat{f}_n(\xi)|^2| \leq 2^{-n}$. Therefore, let

$$\hat{b}_n(\xi) = 2^{-n+2} + |a_n + \hat{f}_n(\xi)|^2.$$

Then, $\hat{b}_n(\xi) \in \mathcal{F}\mathbf{B}$ is a decreasing sequence converging to $c(\xi)$ uniformly. Hence, we have

$$J(c(\xi)|\hat{\phi}(\xi)|^2)) = \lim_{n\to\infty} J(\hat{b}_n(\xi)|\hat{\phi}(\xi)|^2)) \geq 0. \tag{23}$$

This shows that, when ϕ is fixed, the extension is a positive linear functional.

Now suppose that

$$c(\xi)|\hat{\phi}(\xi)|^2) = d(\xi)|\hat{\psi}(\xi)|^2) \geq 0.$$

Since $\hat{b}_n(\xi)|\hat{\phi}(\xi)|^2) > d(\xi)|\hat{\psi}(\xi)|^2)$, we have $J(\hat{b}_n(\xi)|\hat{\phi}(\xi)|^2)) \geq J(d(\xi)|\hat{\psi}(\xi)|^2))$. Hence we have by (23) $J(c(\xi)|\hat{\phi}(\xi)|^2)) \geq J(d(\xi)|\hat{\psi}(\xi)|^2))$.

The converse inequality, the negative and the imaginary parts are treated similarly.

If $k(\xi) \in \mathcal{K}(V)$, then for a sufficiently small neighborhood V of 0 in X the δ-type function $\delta_V \in \mathcal{K}(V)$ satisfies $|\hat{\delta}_V(\xi)|^2 > 0$ on $\operatorname{supp} k$, and hence there is a $c(\xi) \in C(\operatorname{Spec} \mathbf{A})$ such that $k(\xi) = c(\xi)|\hat{\delta}_V(\xi)|^2$. Thus we obtain the following.

Lemma 2. *The positive linear functional J on $\hat{\Lambda}(\hat{X})$ is uniquely extended to a positive linear functional on $\hat{\Lambda}(\hat{X}) + \mathcal{K}(\hat{X})$.*

By the Riesz-Markov-Kakutani theorem there is a measure $d\xi$ such that

$$J(k(\xi)) = \int_{\hat{X}} k(\xi) d\xi \tag{24}$$

for any $k \in \mathcal{K}(\hat{X})$. The translation invariant property (18) holds also for the extended J. Hence $d\xi$ is a Haar measure on $\hat{X}$.

If we show that (24) holds for $k(\xi) \in \hat{\Lambda}(\hat{X})$, then the essential part of the Plancherel theorem follows immediately. In fact, let $\phi(x)$ be an arbitrary function in $L^1(X) \cap L^2(X)$. Then, $k(\xi) = |\hat{\phi}(\xi)|^2 \in \hat{\Lambda}(\hat{X})$ is the Fourier transform of $\phi * \phi^*(x)$ and we have

$$\|\phi\|_{L^2(X)}{}^2 = \phi * \phi^*(0) = J(|\hat{\phi}|^2) = \int |\hat{\phi}(\xi)|^2 d\xi = \|\hat{\phi}\|_{L^2(\hat{X})}{}^2. \tag{25}$$

Proof of (24) *for* $k \in \hat{\Lambda}(\hat{X})$. In view of (21), we may assume without loss of generality that $k(\xi) = |\hat{\phi}(\xi)|^2)$ for a $\phi(x) \in L^1(X) \cap L^2(X)$. Let $h(\xi)$ be an arbitrary function in $\mathcal{K}(\hat{X})$ satisfying $0 \leq h(\xi) \leq 1$. Since J is positive, we have

$$J(|\hat{\phi}|^2) \geq J(h|\hat{\phi}|^2) = \int h(\xi)|\hat{\phi}(\xi)|^2 d\xi,$$

and hence

$$\phi * \phi^*(0) = J(|\hat{\phi}|^2) \geq \int |\hat{\phi}(\xi)|^2 d\xi.$$

Given an $\epsilon > 0$, we can find a neighborhood V of 0 in X such that

$$\begin{aligned} \phi * \phi^*(0) - \epsilon \leq \phi * \phi^* * \delta_V * \delta_V^*(0) &= J(h|\hat{\delta}_V|^2|\hat{\phi}|^2) + J((1-h)|\hat{\delta}_V|^2|\hat{\phi}|^2) \\ &\leq \int |\hat{\phi}(\xi)|^2 d\xi + \|(1-h)|\hat{\delta}_V|^2\|_{C(\mathrm{Spec}\,\mathbf{A})}\|\phi\|^2_{L^2(X)}. \end{aligned}$$

Since $|\hat{\delta}_V(\xi)|^2 \in C_\infty(\hat{X})$, we can make the last term less than ϵ for a sufficiently large h. The discussion in [3] seems obscure in this respect.

Since $L^1(X) \cap L^2(X)$ is dense in $L^2(X)$, the Fourier transformation is extended by continuity to the linear mapping

$$\mathcal{F} : L^2(X) \to L^2(\hat{X}). \tag{26}$$

Theorem 2 (Plancherel). *The Fourier transformation* (26) *is a surjective linear isometry. Its inverse*

$$\overline{\mathcal{F}} : L^2(\hat{X}) \to L^2(X) \tag{27}$$

is obtained by extension by continuity of the inverse Fourier transformation

$$\overline{\mathcal{F}}\hat{f}(x) = \int_{\hat{X}} \hat{f}(\xi)e^{ix\xi}d\xi, \quad \hat{f} \in L^1(\hat{X}) \cap L^2(\hat{X}). \tag{28}$$

Proof. Since $\mathcal{K}(\hat{X})$ is dense in $L^2(\hat{X})$, the surjectivity is proved if we show that for any $k(\xi) \in \mathcal{K}(\hat{X})$ there is a sequence $g_n(x) \in \Lambda(X)$ such that $\|\hat{g}_n - k\|_{L^2(\hat{X})} \to 0$.

We choose, as above, a function $\phi(x) \in L^1(X) \cap L^2(X)$ such that $|\hat{\phi}(\xi)|^2 > 0$ on $\mathrm{supp}\, k$ and write $k(\xi) = c(\xi)|\hat{\phi}(\xi)|^2$ with a continuous function $c(\xi)$. There is a sequence $b_n \in \mathbf{B}$ such that $\hat{b}_n(\xi)$ converges to $c(\xi)$ uniformly on $\mathrm{Spec}\,\mathbf{A}$. Then $g_n = b_n * \phi * \phi^*$ is a desired sequence of functions in $\Lambda(X)$.

Since (26) is an isomorphism of Hilbert spaces, we have

$$\int_X f(x)\overline{g(x)}dx = \int_{\hat{X}} \mathcal{F}f(\xi)\overline{\mathcal{F}g(\xi)}d\xi, \quad f(x), g(x) \in L^2(X). \tag{29}$$

Suppose that $\mathcal{F}f(\xi) \in L^1(\hat{X})$ and that $g(x) \in L^1(X)$. Then, we have by Fubini's theorem

$$\int_X f(x)\overline{g(x)}dx = \int_{\hat{X}} \mathcal{F}f(\xi)d\xi \overline{\int_X g(x)e^{-ix\xi}dx} = \int_X \overline{\mathcal{F}}\mathcal{F}f(x)\overline{g(x)}dx.$$

This shows that

$$f(x) = \overline{\mathcal{F}}\mathcal{F}f(x), \text{ a.e.} x \tag{30}$$

because $L^1(X) \cap L^2(X)$ is dense in $L^2(X)$.

Remark that (19) is the inversion formula (30) for $f(x) \in \Lambda(X)$.

An easy consequence is the following.

Theorem 3 (Pontrjagin). *The character group of $\hat{X}$ is isomorphic to the original group X.*

A function $m(x)$ on X is said to be *positive definite* if it is continuous at the origin and satisfies the condition

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i\overline{\alpha_j}\ m(x_i - x_j) \geq 0, \tag{31}$$

for any $x_j \in X$, and $\alpha_j \in \mathbf{C}$.

It is easily proved that a positive definite function $m(x)$ has the following properties:

(i) $$m(x) = m^*(x) = \overline{m(-x)};$$

(ii) $$|m(x)| \leq m(0);$$

(iii) $m(x)$ is uniformly continuous on X;

(iv) $$\int m(x-y)\phi(y)\overline{\phi(x)}dydx \geq 0, \quad \phi(x) \in L^1(X).$$

Theorem 4 (Bochner). *A function $m(x)$ on X is positive definite if and only if it is the inverse Fourier transform of a finite measure μ on $\hat{X}$:*

$$m(x) = \int_{\hat{X}} e^{ix\xi}d\mu(\xi). \tag{32}$$

proof. The proof of sufficiency is trivial. To prove the necessity, we first assume that $m(x)$ is integrable.

Then, the inequality of property (iv) holds also for any $\phi(x) \in L^2(X)$. That means that the Fourier transform $\hat{m}(\xi) \geq 0$ by the equivalence of (12) and (13). If we choose $\phi(x)$ from $L^1(X) \cap L^2(X)$, then we have $m * \phi * \phi^*(x) \in \Lambda(X)$ and hence

$$\int_{\hat{X}} \hat{m}(\xi)|\hat{\phi}(\xi)|^2 d\xi = m * \phi * \phi^*(0) \leq m(0)\|\phi(x)\|_{L^1(X)}{}^2.$$

Now, we choose the δ-type function $\delta_V(x)$ for $\phi(x)$ and let V tend to the origin. Then we have

$$\int_K \hat{m}(\xi)d\xi \leq m(0)$$

for any compact set $\hat{K}$ in $\hat{X}$. Hence $\hat{m}(\xi)$ is integrable.

Choose a sequence $f_n(x) \in L^1(X) \cap L^2(X)$ tending to $m(x)$ in $L^1(X)$ and let $g(x)$ be an arbitrary function in $\Lambda(X)$. As the limit of equality (29) we have

$$\int_X m(x)\overline{g(x)}dx = \int_{\hat{X}} \mathcal{F}m(\xi)\overline{\mathcal{F}g(\xi)}d\xi = \int_X \overline{\mathcal{F}}\mathcal{F}m(x)\overline{g(x)}dx, \quad g(x) \in \Lambda(X),$$

and hence

$$m(x) = \int_{\hat{X}} e^{ix\xi}\, \hat{m}(\xi)d\xi.$$

In the general case, we note that products of positive definite functions are positive definite. Let $\hat{V}$ be a compact neighborhood of 0 in $\hat{X}$ and consider the positive definite function

$$m_{\hat{V}}(x) = m(x)\left| |\hat{V}|^{-1} \int_{\hat{V}} e^{ix\xi}d\xi \right|^2.$$

This is integrable because $m(x)$ is bounded and the inverse Fourier transform of the characteristic function of $\hat{V}$ is square integrable. As $\hat{V}$ shrinks to 0, $m_{\hat{V}}(x)$ converges to $m(x)$ uniformly on compact sets. From the above proof we have

$$m_{\hat{V}}(x) = \int_{\hat{X}} e^{ix\xi}\, \hat{m}_{\hat{V}}(\xi)d\xi$$

with an integrable function $\hat{m}_{\hat{V}}(\xi) \geq 0$ and

$$\int_{\hat{X}} \hat{m}_{\hat{V}}(\xi)d\xi = m_{\hat{V}}(0) = m(0).$$

Thus the set $\{\hat{m}_{\hat{V}}(\xi)d\xi\}$ of measures is relatively weak*-compact in the dual of $C_\infty(\hat{X})$. Let the measure μ be an accumulation point. Then we have (32).

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