SMOOTHING EFFECTS FOR NONLINEAR DERIVATIVE SCHRÖDINGER EQUATIONS

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1. Introduction

This presentation deals with the study of smoothing and analyticity effects for some nonlinear derivative Schrödinger equations.

2. Smoothing effects for some derivative nonlinear Schrödinger equations

This a joint work with Pr. Nakao Hayashi and Pr. Pavel.I Naumkin. In this part we study a smoothing property of solutions to the Cauchy problem for the nonlinear Schrödinger equations :

$$\begin{cases} \imath u_t + u_{xx} = \mathcal{N}(u, \overline{u}, u_x, \overline{u_x}), & x \in \mathbb{R}, \quad t \in \mathbb{R}, \\ u(x,0) = u_0(x) \ , & x \in \mathbb{R}, \end{cases} \tag{1}$$

where the nonlinearity is

$$\mathcal{N}(u, \overline{u}, u_x, \overline{u_x}) = K_1|u|^2 u + K_2 |u|^2 u_x + K_3 u^2 \overline{u_x} + K_4 |u_x|^2 u + K_5 \overline{u} u_x^2 + K_6 |u_x|^2 u_x.$$

The function $K_j = K_j(|u|^2)$ satisfy $K_j(z) \in \mathcal{C}^{l+3}([0,+\infty);\mathbb{C})$. If $K_5(z) = \frac{1}{1+z}$ and $K_1 = K_2 = K_3 = K_4 = K_6 = 0$ then the equation (1) is called classical pseudospin magnet model [5]. We introduce some functionnal spaces in order to state our results : $\mathcal{L}^p(\mathbb{R}) = \{\varphi \in \mathcal{S}'(\mathbb{R}) : \|\varphi\|_p < \infty\}$, where $\|\varphi\|_p = \left(\int |\varphi(x)|^p dx\right)^{1/p}$ if $1 \leq p < \infty$, if $p = \infty$ then $\|\varphi\|_\infty = \text{ess.sup}\{|\varphi(x)|; x \in \mathbb{R}\}$. In order to keep the notations clear, we write $\|\varphi\| = \|\varphi\|_2$. The weighted Sobolev spaces are : $\mathcal{H}_p^{m,s} = \{\varphi \in \mathcal{S}'(\mathbb{R}) : \|\varphi\|_{m,s,p} = \left\| \left(1+x^2\right)^{s/2} \left(1-\partial_x^2\right)^{m/2} \varphi \right\|_p < \infty\}$, $m, s \in \mathbb{R}$, $1 \leq p \leq \infty$. For simplicity, for $p = 2$ we write $\mathcal{H}^{m,s} = \mathcal{H}_2^{m,s}$ and $\|\phi\|_{m,s} = \|\phi\|_{m,s,2}$. We denote also $\mathcal{H}^{m,\infty} = \cap_{s=1}^{\infty} \mathcal{H}^{m,s}$. Let $\mathcal{C}(\mathcal{I};\mathcal{B})$ be the space of continuous functions from a time interval $\mathcal{I}$ to a Banach space $\mathcal{B}$. Our main results of this section are the followings.

Theorem 1. *We assume that the nonlinear term $\mathcal{N}$ does not depend on $\bar{u}_x$ and the initial data u_0 is such that $u_0 \in \mathcal{H}^{3,l}$ with $l \in \mathbb{N}$. Then there exists $T > 0$ such that there exists a unique solution*

$$u \in \mathcal{C}\left([-T,T];\mathcal{H}^{2,0}\right) \cap \mathcal{L}^\infty\left(-T,T;\mathcal{H}^{3,0}\right) \cap \mathcal{C}\left([-T,T]\backslash\{0\};\mathcal{C}^{l+2}(\mathbb{R})\right)$$

of the Cauchy problem (1) such that

$$\sup_{t\in[-T,T]} |t|^k \left\| \left(1+x^2\right)^{-k/2} \partial_x^k u(t) \right\|_{2,0} < \infty \ \textit{for} \quad 0 \leq k \leq l.$$

For the case of nonlinearities depending on $\bar{u}_x$, we have to assume the smallness condition on the initial data. We have the following result.

Theorem 2. *We assume that the nonlinear term $\mathcal{N}$ depends on $\bar{u}_x$ and that the initial data u_0 is such that $u_0 \in \mathcal{H}^{3,l}$ with $l \in \mathbb{N}$ and $\|u_0\|_{3,l}$ is small enough . Then there exists $T > 0$ such that there exists a unique solution which has the same results as in Theorem 1*

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In this work, we apply an operator which is defined by N. Hayashi, P.I. Naumkin and the author [3]. This operator is defined by $\mathcal{S}(\varphi) = \cosh(\varphi) + i\sinh(\varphi)\mathcal{H}$. φ is defined later as

$$\varphi(x,t) = \frac{1}{\delta}\sum_{k=0}^{l}\partial_x^{-1}\left(|\mathcal{J}^k\tilde{u}(x,t)|^2 + |\mathcal{J}^k\tilde{u}_x(x,t)|^2 + \sum_{2\le j\le 5}|\mathcal{J}^k K_j\tilde{u}|^2 + |\mathcal{J}^k K_6\tilde{u}_x|^2\right),$$

where $\mathcal{H}\phi = \frac{1}{\pi}\mathrm{Pv}\int\frac{\phi(x')dx'}{x-x'}$, $\partial_x^{-1} = \int_{-\infty}^{x}\dots dx'$ and $\tilde{u}$ is related to solutions to (1). It enables us to avoid the use of the well known results of the pseudo-differential operators and so by virtue of simple explicit computations we treat the problem in the natural order Sobolev space $\mathcal{H}^{3,0}$.

3. Smoothing effects for some derivative nonlinear Schrödinger equation without smallness condition

We consider the equation described in the previous part and we will remove the smallness condition on the data assumed in Theorem 1. To obtain the new result, we combine a diagonalization technique to the method used in part 1.

Theorem 3. *We assume that the initial data u_0 is such that $u_0 \in \mathcal{H}^{3,l}$ with $l \in \mathbb{N}$. Then there exists $T > 0$ such that there exists a unique solution*

$$u \in \mathcal{C}\left([-T,T];\mathcal{H}^{2,0}\right) \cap \mathcal{L}^\infty\left(-T,T;\mathcal{H}^{3,0}\right) \cap \mathcal{C}\left([-T,T]\backslash\{0\};\mathcal{C}^{l+2}(\mathbb{R})\right)$$

of the Cauchy problem (1) such that

$$\sup_{t\in[-T,T]}|t|^k\left\|\left(1+x^2\right)^{-k/2}\partial_x^k u(t)\right\|_{3,0} < \infty \text{ for } \quad 0 \le k \le l.$$

Note that there is no more a any smallness condition on the initial data.

4. Analytical smoothing effects for some derivative nonlinear Schrödinger equations

This is a joint work with Pr. Nakao Hayashi and Pavel I. Naumkin. We study an analytical smoothing effect of solutions to the Cauchy problem,

$$\begin{cases} iu_t + u_{xx} = \mathcal{N}(u,\bar{u},u_x,\bar{u}_x), & x\in\mathbb{R}, \quad t\in\mathbb{R}, \\ u(0,x) = u_0(x), & x\in\mathbb{R}, \end{cases} \tag{2}$$

where the nonlinearity is

$$\mathcal{N}(u,\bar{u},u_x,\bar{u}_x) = K_1|u|^2u + K_2|u|^2u_x + K_3u^2\bar{u}_x + K_4|u_x|^2u + K_5\bar{u}u_x^2 + K_6|u_x|^2u_x,$$

The coefficients $K_j = K_j(|u|^2) = K_j(f)$ are analytic and have analytic continuations $z = f + ig$ in the circle $|z| < \rho$. Thus these functions can be written with their Taylor expansion

$$K_j(z) = \sum_{n=0}^{\infty}a_{n,j}z^n, \qquad a_{n,j} = \frac{1}{n!}K_j^{(n)}(0) = \frac{1}{n!}\frac{d^n}{dz^n}K_j(0).$$

We also assume that $\sum_{n=0}^{\infty}|a_{n,j}||z|^n \le C(\rho)$ for $|z| < \rho$, where $C(\rho)$ is a continuous function with ρ as a variable. Our aim is to extend the results described in the first and second part and to the analytical case and to find a solution of (2) and study analytical problems. Precisely if the initial data u_0 satisfy the condition $(\cosh\beta x)u_0 \in \mathcal{H}^{3,0}$ with the decay $||(\cosh\beta x)u_0||_{3,0} < \rho$ in case of a nonlinearity $\mathcal{N}$ which does not depend on the term $\bar{u}_x$, then there exists a unique solution u of the Cauchy problem (2) and a time $T > 0$ which is analytic with respect to x and has an analytic continuation on the complex plane $z = x + iy$ with $|y| < 2|\beta t|$, for all $t \in [-T,T]\backslash\{0\}$. In case of a nonlinearity $\mathcal{N}$ depending on $\bar{u}_x$, we must assume again the additional

smallness condition on the initial data u_0.
So in this part we extend the estimations proved for the linear Schrödinger in the first section to the analytical case. Then the result stated above is proved by a contraction mapping method. We sum up these results in the following theorems.

Theorem 4. *We assume that the nonlinear term $\mathcal{N}$ does not depend on $\bar{u}_x$, and the initial data u_0 are such that $u_0 \cosh \beta x \in \mathcal{H}^{3,0}$, where $\beta \in \mathbb{R}$ and the norm $||u_0 \cosh \beta x||_{3,0} < \rho$. Then for some time $T > 0$ there exists a unique solution u of the Cauchy problem (1) such that $u \in \mathcal{C}\left([-T,T];\mathcal{H}^{2,0}\right) \cap \mathcal{L}^\infty\left(-T,T;\mathcal{H}^{3,0}\right)$ and the solution u has an analytic continuation $u(t,z)$ to the strip $\left\{z = x+iy; \quad -\infty < x < \infty, -2|t\beta| < y < 2|t\beta|, \quad t \in [-T,T]\backslash\{0\}\right\}$ satisfying the estimate*

$$\sup_{-2|t\beta|<y<2|t\beta|} |u(t,x+iy)| \leq C \cosh \beta x ||u_0 \cosh \beta x||_{3,0} \quad \text{for all} \quad (t,x) \in [-T,T]\backslash\{0\} \times \mathbb{R}.$$

For the case of the nonlinearities depending on $\bar{u}_x$ we have to assume the additional smallness condition on the initial data. We prove the following result.

Theorem 5. *We assume that the nonlinear term $\mathcal{N}$ depends on $\bar{u}_x$, the initial data u_0 are such that $u_0 \cosh \beta x \in \mathcal{H}^{3,0}$, $\beta \in \mathbb{R}$ and the norm $||u_0 \cosh \beta x||_{3,0}$ is sufficiently small. Then the same results as in Theorem 4 are true.*

Through our recent research, we are able to prove an improvement of the previous theorem in this way :

Theorem 6. *We assume that the initial data u_0 are such that $u_0 \cosh \beta x \in \mathcal{H}^{3,0}$, $\beta \in \mathbb{R}$. Then the same results as in Theorem 4 are true.*

Therefore in this theorem, we do not assume anymore any smallness condition on the initial data. Therefore we improve theorem 5.

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