Dynamic portfolio selection with maximum absolute deviation model

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Abstract

In this paper, we present a new multiperiod portfolio selection model with maximum absolute deviation model. The investor is assumed to seek an investment strategy to maximize his/her terminal wealth, and minimize the total risk in all periods. Different from original consideration that risk is defined as the variance of terminal wealth, in our paper, the total risk is defined as the average of the sum of maximum absolute deviation of all assets in all periods. At the same time, noticing that the risk during the period is so high that the investor may go bankrupt, a maximum risk level is given in every period. By introducing an auxiliary problem, we deduce the optimal strategy via the dynamic programming method.

Key Words: Portfolio optimization, dynamic programming, maximum absolute deviation.

1.INTRODUCTION

Multiperiod portfolio selection model has attracted much attention in the financial field because a portfolio is usually rebalanced dynamically in order to get a desired return even if the situation changes in the future. Hence, how to extend the single period model to dynamic portfolio situation is always a hot topic. For

Till now, many possible definitions of risk have been proposed in the literature in the several decades because different investors adopt different investment strategies in seeking to realize their investment objectives. Thus, there is no unique risk measure exists can be used to solve every investor’s problem. Konno and Yamazaki (1990, 1991) propose a new risk measure method, mean absolute deviation model (MAD), as an alternative to constructing optimal strategy. Compared to classical mean variance method, the portfolio model with MAD can be finally transformed into linear programming problem. Simplicity and computational robustness are perceived as one of the most important advantages of the MAD model (see Konno and Shirakawa (1994), Konno and Wijayanayake (1999, 2001a, 2001b, 2002)). Much attention has been focused in this model. One of the important topic is to present some new risk-control models based on MAD. For example, Cai et al. (2000) introduce the maximum absolute deviation risk model $l_{\infty}$ as follows:

$$w_{\infty}(x) = \max_{1 \leq j \leq n} E |R_j x_j - E(R_j) x_j| = \max_{1 \leq j \leq n} E |R_j x_j - r_j x_j|$$

The portfolio problem is defined as the following:

$$\begin{cases} 
\min & (\max_{1 \leq j \leq n} q_j x_j, - \sum_{j=1}^{n} r_j x_j) \\
\text{s.t.} & \sum_{j=1}^{n} x_j = M_0 \\
& x_j \geq 0, j = 1, \ldots, n 
\end{cases}$$

where $q_i = E|R_i - r_i|$, $i = 1, \ldots, n$, $M_0$ is the initial wealth the investor has. In this model, the investor is assumed to minimize the maximum of individual risk. Under the assumption that $r_1 \leq r_2 \leq \cdots \leq r_n$ and $q_i > 0$ for $i = 1, \ldots, n$, they give the explicit analytical solution for the model and the entire efficient frontier is
also plotted. It is also pointed out that such a risk model is very conservative and it does not explicitly involve the covariance of the asset returns.

The alternative \( l_\infty \) risk function proposed by Teo and Yang (2001) is as follows:

\[
H_{\infty}^T(x) = \frac{1}{T} \sum_{t=1}^{T} \max_{1 \leq i \leq n} E \left| R_{it}x_i - r_{it}x_i \right|,
\]

where \( R_{it} \) is random variables and \( r_{it} \) is the expected value of \( R_{it} \), for \( i = 1, \ldots, n, \ t = 1, \ldots, T \). This function is an extension of \( l_\infty(x) \), and it is assumed that the available historical data are split into \( T \) periods. In each period, the individual absolute deviation with respect to the expected value of the period is calculated. The total risk of the portfolio is taken as the average of the maximum (over all assets) of these individual absolute deviations over all periods.

Although the application of MAD or those linear models based on it is successful in portfolio theory, there is little literature about extending these linear models to multiperiod case. Hence, in this paper, we try to consider a multiperiod portfolio selection with \( l_\infty \) model. Different with the classical M-V model, we employ the \( l_\infty \) function to control the risk in every period. We assume that the investor wants to maximize the total wealth at the end of investment period and minimize the risk which is defined as the average of the total risk in all periods. The risk in every period is defined as the maximum absolute deviation of all assets. Moreover, considering that if the risk during the period is so high that the investor can not finish the whole investment period, we employ a parameter to control the risk in every period. That is, the risk in every period can not be above the given risk level. Such a consideration make the problem more complicated, but at the same time, more reasonable for the investor. By using dynamic programming method, we deduce the closed form solution.

The organization of this paper is as this: In Section 2, some important notations are introduced and the basic model are set up for the financial market. In Section 3, the process of solving the optimal strategy for the investment model is presented. First, a auxiliary problem is employed and then we prove how to obtain the final optimal strategy via the solution to auxiliary problem. The conclusion is given in
Section 4 and the proof of main theorems can be found in Appendix.

2. NOTATION AND MODEL

We consider a capital market with $n$ risky assets $S_j$, $j = 1, \cdots, n$, whose return rate are random. An investor is assumed to allocate his initial wealth denoted by $W_0$ among the $n$ risky assets in the beginning of the 1th period and get the final wealth at the end of the $T$th period. It is a dynamic investment selection, i.e. the wealth can be reallocated among the $n$ risky assets at the beginning of each of the following $T - 1$ consecutive time periods.

To make the following discussion conveniently, we will rearrange the $n$ assets in order that all assets is ranked by the value of expected return rate in every period.

Let $e_t = (e_{t1}, \cdots, e_{tn})'$, where $e_{tj}$ is the return rate of asset $S_j$ in period $t$, $t = 1, \cdots, T$, which is a random variable. It is assumed that vectors $e_t$, $t = 1, \cdots, T$, are statistically independent and return $e_t$ has a known mean $E(e_t) = (E(e_{t1}), \cdots, E(e_{tn}))'$.

In the 1th period, we assume that $Ee_{t1} \leq Ee_{t2} \cdots \leq Ee_{tn}$. Let $I_1 = \{S_1, \cdots, S_n\}$. Then in the following $t$ periods, $t = 2, \cdots, T$, we rearrange the order of $n$ assets in $I_1$ and obtain $I_t = \{S_{k_t1}, \cdots, S_{kn_t}\}$, where $k_ti \in \{1, \cdots, n\}$, and $k_{ti} \neq k_{jt}$ for $i \neq j$, such that

$$Ee_{t{k_t1}} \leq Ee_{t{k_t2}} \cdots \leq Ee_{t{k_tn_t}}$$

Let $k_t1 = 1, \cdots, k_{tn_t} = n$. Let $R_{tj}$ be the return rate of asset $S_{k_tj}$, $t = 1, \cdots, T$, $j = 1, \cdots, n$, and $r_{tj}$ the expected value of $R_{tj}$. Then $R_{tj} = e_{t{k_tj}}$, and $r_{tj} = Ee_{t{k_tj}}$. Clearly

$$r_{t1} \leq r_{t2} \cdots \leq r_{tn}$$

For $t = 1, \cdots, T$, denote by $R_t = (R_{t1}, \cdots, R_{tn})'$, and $r_t = (r_{t1}, \cdots, r_{tn})'$.

Denote by $x_t = (x_{t1}, \cdots, x_{tn})'$, where $x_{tj}$ is the amount invested in the asset $S_{k_tj}$ at the beginning of the $t$th time period.

Denote by $V_t$ the total wealth the investor obtains at the end of the $t$th period. Let $V_0 = W_0$. Clearly,

$$V_t = V_{t-1} + R'_tx_t, \quad t = 1, \cdots, T$$
It is assumed that the whole investment is a self-financing process. The investor does not increase the money or put aside some money in the whole period, i.e., the amount of money allocated to every asset in the $t$th period is equal to the total wealth at the end of the $t-1$th period. That is,

\[(2.2) \sum_{j=1}^{n} x_{tj} = V_{t-1}, \quad t = 1, \ldots, T\]

In the following discussion, we focus on how to control the risk in every period.

We employ the $l_\infty$ risk function to measure the risk in $t$th period which is denoted by $w_t(x_t)$, then we obtain

\[w_t(x_t) = \max_{1 \leq j \leq n} E(|R_{tj}x_{tj} - r_{tj}x_{tj}|), \quad t = 1, \ldots, T\]

Denote by $w'_t$ the total risk at the end of $t$ periods, which is defined as follows:

\[w'_0 = 0, \quad w'_t = w'_{t-1} + \max_{1 \leq j \leq n} E(|R_{tj}x_{tj} - r_{tj}x_{tj}|), \quad t = 1, \ldots, T.\]

It is assumed that $\varepsilon_t$ is the maximum risk level in period $t$, $t = 1, \ldots, T$. We assume that the risk in period $t$ can not be above $\varepsilon_t V_{t-1}$, i.e.,

\[(2.4) \max_{1 \leq j \leq n} E(|R_{tj}x_{tj} - r_{tj}x_{tj}|) \leq \varepsilon_t V_{t-1}, \quad t = 1, \ldots, T\]

It is assumed that short selling is not allowed:

\[(2.5) \quad x_{tj} \geq 0, \quad t = 1, \ldots, T, \quad j = 1, \ldots, n\]

\[
\begin{aligned}
\min & \frac{\lambda}{T} \sum_{t=1}^{T} \max_{1 \leq j \leq n} E|R_{tj}x_{tj} - r_{tj}x_{tj}| - (1 - \lambda)E(V_T) \\
\text{s.t.} & V_t = V_{t-1} + R_t x_t \\
& \max_{1 \leq j \leq n} E|R_{tj}x_{tj} - r_{tj}x_{tj}| \leq \varepsilon_t V_{t-1}, \quad t = 1, \ldots, T \\
& \sum_{j=1}^{n} x_{tj} = V_{t-1}, \quad t = 1, \ldots, T \\
& x_{tj} \geq 0, \quad j = 1, \ldots, n, \quad t = 1, \ldots, T
\end{aligned}
\]

We assume that the investor is risk averse. He/she wants to maximize the terminal wealth in the final period, i.e., $EV_T$; on the other hand, he/she wants to minimize the average value of total risk in $T$ periods, i.e., $\frac{1}{T}w'_T$. Thus, our portfolio selection problem can be formulated as the following programming problem, which is denoted by $P_1$:

\[
\begin{aligned}
\min & \frac{\lambda}{T}w'_T - (1 - \lambda)E(V_T) \\
\text{s.t.} & (2.1) - (2.5)
\end{aligned}
\]
Here $\lambda \in (0, 1)$ can be considered as the risk preference of the investor. The greater $\lambda$ is, more conservative the investor is.

3. OPTIMAL STRATEGY FOR THE PORTFOLIO MODEL

In this section, we will derive the analytical solution to $P_1$ via dynamic programming method. A numerical example is also given to demonstrate the application of the model.

First, let’s introduce the following problem.

Define $P_2$ as follows:

$$
\begin{align*}
\min & \quad \frac{\lambda}{T} w_T^1 - (1 - \lambda) E(V_T) \\
\text{s.t.} & \quad V_t = V_{t-1} + R_t x_t \\
& \quad w_t^1 = w_{t-1}^1 + z_t, \quad t = 1, \ldots, T \\
& \quad E[R_{tj} x_{tj} - r_{tj} x_{tj}] \leq z_t, \quad j = 1, \ldots, n, t = 1, \ldots, T \\
& \quad E[R_{tj} x_{tj} - r_{tj} x_{tj}] \leq \varepsilon_t V_{t-1}, \quad j = 1, \ldots, n, t = 1, \ldots, T \\
& \quad \sum_{j=1}^n x_{tj} = V_{t-1}, \quad t = 1, \ldots, T \\
& \quad x_{tj} \geq 0, \quad j = 1, \ldots, n, t = 1, \ldots, T
\end{align*}
$$

with $w_0^1 = 0, V_0 = W_0$.

The following theorem set up the relationship between $P_1$ and $P_2$.

**Theorem 3.1** (1) If $X = (x_1, \ldots, x_T)'$ is an optimal solution to $P_1$, then $(X, Z)$ is an optimal solution to $P_2$, where $Z = (z_1, \ldots, z_T)'$, $z_t = \max_{1 \leq j \leq n} E[R_{tj} x_{tj} - r_{tj} x_{tj}]$.

(2) If $(X, Z)$ is an optimal solution to $P_2$, then $X$ is an optimal solution to $P_1$, where $X = (x_1, \ldots, x_T)'$.

**Proof.** (1) If $X$ is an optimal solution to $P_1$, then $(X, Z)$ is an optimal solution to $P_2$, where $Z = (z_1, \ldots, z_T)'$ and $z_t = \max_{1 \leq j \leq n} E[R_{tj} x_{tj} - r_{tj} x_{tj}]$. Otherwise, there exists a feasible solution $(Y, Z^1)$ to $P_2$, where $Y = (y_1, \ldots, y_T)'$, $Z^1 = (z_1^1, \ldots, z_T^1)$, satisfying that

$$
\begin{align*}
V_0^1 &= W_0, V_t^1 = V_{t-1}^1 + R_t' y_t, t = 1, \ldots, T \\
w_T^1 &= \sum_{t=1}^T z_t^1 \\
E[R_{tj} y_{tj} - r_{tj} y_{tj}] &\leq z_t^1, \quad j = 1, \ldots, n, t = 1, \ldots, T
\end{align*}
$$
Denote by $\lambda_1$ such that

$$y_{tj} \geq 0, \; j = 1, \ldots, n; \; t = 1, \ldots, T$$

$$\sum_{j=1}^{n} y_{tj} = V_{t-1}^1, \; t = 1, \ldots, T.$$ 

such that

$$\frac{\lambda}{T} w_T^1 - (1-\lambda)E(V_1^1) < \frac{\lambda}{T} w_T^1 - (1-\lambda)E(V_T^1) = \frac{\lambda}{T} \sum_{t=1}^{T} \max_{1 \leq j \leq n} E[R_{tj}y_{tj} - r_{tj}y_{tj}] - (1-\lambda)E(V_T^1).$$

Obviously, $Y$ is also a feasible solution to $P_1$ and

$$\frac{\lambda}{T} \sum_{t=1}^{T} \max_{1 \leq j \leq n} E[R_{tj}y_{tj} - r_{tj}y_{tj}] - (1-\lambda)E(V_T^1) \leq \frac{\lambda}{T} w_T^1 - (1-\lambda)E(V_T^1)$$

which contradicts that $X$ is an optimal solution to $P_1$.

(ii) If $(X, Z)$ is an optimal solution to $P_2$, then $X = (x_1, \ldots, x_T)'$ is an optimal solution to $P_1$. Otherwise, there exists a feasible solution $Y = (y_1, \ldots, y_T)'$ to $P_1$, satisfying that

$$V_0' = 0, \; V_t' = V_{t-1}' + R_t'y_t, \; t = 1, \ldots, T$$

$$\max_{1 \leq j \leq n} E[R_{tj}y_{tj} - r_{tj}y_{tj}] \leq \varepsilon_t, \; t = 1, \ldots, T$$

$$\sum_{j=1}^{n} y_{tj} = V_{t-1}'^1, \; t = 1, \ldots, T.$$ 

$$y_{tj} \geq 0, \; j = 1, \ldots, n; \; t = 1, \ldots, T$$

such that

$$\frac{\lambda}{T} \sum_{t=1}^{T} \max_{1 \leq j \leq n} E[R_{tj}y_{tj} - r_{tj}y_{tj}] - (1-\lambda)E(V_T^1) < \frac{\lambda}{T} \sum_{t=1}^{T} \max_{1 \leq j \leq n} E[R_{tj}x_{tj} - r_{tj}x_{tj}] - (1-\lambda)E(V_T^1)$$

Denote by $z_t^1 = \max_{1 \leq j \leq n} E[R_{tj}y_{tj} - r_{tj}y_{tj}], \; z_t = \max_{1 \leq j \leq n} E[R_{tj}x_{tj} - r_{tj}x_{tj}], \; w_t^' = \sum_{t=1}^{T} z_t, \; w_T^1 = \sum_{t=1}^{T} z_t^1$, then $(Y, z_T^1)$ is also a feasible solution to $P_2$ and

$$\frac{\lambda}{T} w_T^1 - (1-\lambda)E(V_T^1) < \frac{\lambda}{T} w_T^1 - (1-\lambda)E(V_T^1)$$

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which contradicts that \((X, Z)\) is an optimal solution to \(P_2\). We complete the proof.

Now we use the dynamic programming method to solve the problem \(P_2\). Denote by
\[
f_T(V_T, w_T^1) = \frac{\lambda}{T} w_T^1 - (1 - \lambda) E(V_T),
\]
and
\[
f_{t-1}(V_{t-1}, w_{t-1}^1) = \min_{(x_t, z_t)} E[f_t(V_t, w_t^1)|(V_{t-1}, w_{t-1}^1)], t = 1, \cdots, T
\]
First we consider the \(T\)th period by assuming that \((V_{T-1}, w_{T-1}^1)\) is known. Then
\[
f_{T-1}(V_{T-1}, w_{T-1}^1) = \min_{(x_{T}, z_{T})} E[(V_T, w_T^1)|(V_{T-1}, w_{T-1}^1)]
\]
\[
= \min_{(x_{T}, z_{T})} \lambda_T (w_{T-1}^1 + z_T) - (1 - \lambda_T) (V_{T-1} + \sum_{j=1}^{n} r_T j x_T)
\]
The problem \(P_2\) can be transformed as the following problem denoted by \(P_T\)
\[
\begin{align*}
\min & \quad \{ \lambda_T (w_{T-1}^1 + z_T) - (1 - \lambda_T) (V_{T-1} + \sum_{j=1}^{n} r_T j x_T) \} \\
\text{s.t} & \quad q_{Tj} x_T \leq z_T, j = 1, \cdots, n \\
& \quad q_{Tj} x_T \leq \varepsilon_T V_{T-1}, j = 1, \cdots, n \\
& \quad \sum_{j=1}^{n} x_T = V_{T-1} \\
& \quad x_T \geq 0, j = 1, \cdots, n
\end{align*}
\]
where \(q_{Tj} = E[R_{Tj} - r_{Tj}], j = 1, \cdots, n\).

To solve this problem, first let us introduce an auxiliary problem. Denote \(P_T'\) as follows:
\[
\begin{align*}
\min & \quad \{ \lambda_T (w_{T-1}^1 + z_T) - (1 - \lambda_T) (V_{T-1} + \sum_{j=1}^{n} r_T j x_T) \} \\
\text{s.t} & \quad q_{Tj} x_T \leq z_T, j = 1, \cdots, n \\
& \quad \sum_{j=1}^{n} x_T = V_{T-1} \\
& \quad x_T \geq 0, j = 1, \cdots, n
\end{align*}
\]
The difference between \(P_T\) and \(P_T'\) is that constraints \(q_{Tj} x_T \leq \varepsilon_T V_{T-1}\) is omitted in \(P_T'\). The optimal solution to this problem has been obtained in Cai et al.(2000). We denote it as \((x_T^*, z_T^*)\), where \(x_T^* = (x_{T1}^*, \cdots, x_{Tn}^*)\). Then we will prove that the optimal solution to \(P_T\) denoted by \((x_T^{**}, z_T^{**})\) can be obtained by \((x_T^*, z_T^*)\) under some assumptions.
Lemma 3.1 The solution to $P'_T$ is as follows:

\[ x^*_{Tj} = \begin{cases} \frac{V_{T-1}}{q_{Tj}} \left( \sum_{i \in A_T(\lambda)} \frac{1}{q_{Ti}} \right)^{-1}, & j \in A_T(\lambda) \\ 0, & j \not\in A_T(\lambda) \end{cases} \]

\[ z^*_T = V_{T-1} \left( \sum_{i \in A_T(\lambda)} \frac{1}{q_{Ti}} \right)^{-1} \]

where $A_T(\lambda)$ is determined by:

1) If there exists an integer $p_T \in [0, n - 2]$ such that

\[ \frac{r_{Tn} - r_{Tn-1}}{q_{Tn}} < \frac{\lambda}{1 - \lambda} \]

\[ \frac{r_{Tn} - r_{Tn-2}}{q_{Tn}} + \frac{r_{Tn-1} - r_{Tn-2}}{q_{Tn-1}} < \frac{\lambda}{1 - \lambda} \]

\[ \frac{r_{Tn} - r_{Tn - p_T}}{q_{Tn}} + \frac{r_{Tn-1} - r_{Tn - p_T}}{q_{Tn-1}} + \cdots + \frac{r_{Tn-p_T+1} - r_{Tn-p_T}}{q_{Tn-p_T+1}} < \frac{\lambda}{1 - \lambda} \]

and

\[ \frac{r_{Tn} - r_{Tn - p_T - 1}}{q_{Tn}} + \frac{r_{Tn-1} - r_{Tn - p_T - 1}}{q_{Tn-1}} + \cdots + \frac{r_{Tn-p_T - 1} - r_{Tn-p_T}}{q_{Tn-p_T}} \geq \frac{\lambda}{1 - \lambda} \]

then

\[ A_T(\lambda) = \{n, n - 1, \cdots, n - p_T\} \]

2) Otherwise, if the above condition is not satisfied by any integer $p_T \in [0, n - 2]$, then

\[ A_T(\lambda) = \{n, n - 1, \cdots, 1\} \]

Here, in order to make the following description simplified, we denote $A_T(\lambda) = \{1, \ldots, k_T\}$, $k_T \leq n$. It is necessary to notice that the rank in $A_T(\lambda)$ now is converse to the original one. That is, for $i, j \in \{n, n - 1, \ldots, n - p_T\}$ or $\{n, n - 1, \cdots, 1\}$, with $i < j$, we have $r_i \leq r_j$. But for $i, j \in \{1, \ldots, k_T\}$, with $i < j$, we have $r_i \geq r_j$.

Now we consider the solution to $P_T$. It is easy to know that if $z^*_T \leq \varepsilon_T$, then $(x^*_T, z^*_T)$ is also the optimal solution to $P_T$. If $z^*_T > \varepsilon_T$, then the feasible solution of $P_T$ may be 0 if $\varepsilon_T \to 0$. This means that the investor is so conservative that do not want to bear any risk. The following theorem give the optimal solution to $P_T$:
Theorem 3.2 The optimal solution to $P_T$ is as follows:

1) If $z_T^* \leq \varepsilon_T V_{T-1}$, then $(x_T^*, z_T^*)$ is also an optimal solution to $P_T$;

2) If $z_T^* > \varepsilon_T V_{T-1}$, then the optimal solution to $P_T$ is as follows:

If there exists $n \geq l_T > k_T$, such that

$$\frac{k_T}{\sum_{j=1}^{k_T} \frac{1}{q_T j}} > \frac{\varepsilon_T V_{T-1}}{z_T^*} \geq \frac{l_T}{\sum_{j=1}^{l_T} \frac{1}{q_T j}},$$

then the optimal solution to $P_T$, denoted by $(x_T^{**}, z_T^{**})$, where $x_T^{**} = (x_{T1}^{**}, \ldots, x_{Tn}^{**})'$ is as follows:

$$x_T^{**} = \frac{\varepsilon_T}{q_T} V_{T-1}, j = 1, \ldots, l_T - 1$$

$$x_T^{**} = z_T^* \frac{k_T}{\sum_{j=1}^{k_T} \frac{1}{q_T j}} - \varepsilon_T V_{T-1} \sum_{j=1}^{l_T-1} \frac{1}{q_T j}$$

$$x_T^{**} = 0, j = l_T + 1, \ldots, n$$

$$z_T^{**} = \varepsilon_T V_{T-1}$$

The proof of this theorem can be found in Appendix.

Now, we substitute $(x_T^{**}, z_T^{**})$ to $f_{T-1}$.

If $z_T^* \leq \varepsilon_T V_{T-1}$, then by Theorem 1, we have

$$f_{T-1}(V_{T-1}, w_{T-1}^1) = \min_{(x_T, z_T)} E[f_T(V_T, w_T^1)|(V_{T-1}, w_{T-1}^1)]$$

(3.1)

$$= \frac{\lambda}{T} (w_{T-1}^1 + z_T^*) - (1 - \lambda)[V_{T-1} + \sum_{j=1}^n r_{Tj} x_T^{**}]$$

$$= \frac{\lambda}{T} w_{T-1}^1 - c_T V_{T-1}$$

where

$$c_T = (1 - \lambda)(1 + a_T b_T) - \frac{\lambda}{T} a_T, a_T = \left(\sum_{j=1}^{k_T} \frac{1}{q_T j}\right)^{-1}, b_T = \sum_{j=1}^{k_T} \frac{r_{Tj}}{q_T j}$$

Then in this case,

$$f_{T-2}(V_{T-2}, w_{T-2}^1) = \min_{(x_{T-1}, z_{T-1})} E[f_{T-1}(V_{T-1}, w_{T-1}^1)|(V_{T-2}, w_{T-2}^1)]$$

(3.2)

$$= \min_{(x_{T-1}, z_{T-1})} \left\{ \frac{\lambda}{T} [w_{T-2}^1 + z_{T-1}] - c_T [V_{T-2} + \sum_{j=1}^n r_{T-1j} x_{T-1j}^*] \right\}$$

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If \( z^*_T > \varepsilon_T \),
\[
f_{T-1}(V_{T-1}, w^1_{T-1}) = \min_{(x_T, \varepsilon_T)} E[f_T(V_T, w_T)|(V_{T-1}, w^1_{T-1})]
\]
\[
= \frac{\lambda}{T} (w^1_{T-1} + z^*_T) - (1 - \lambda)[V_{T-1} + \sum_{j=1}^n r_T^j x^*_T]
\]
\[
= \frac{\lambda}{T} (w^1_{T-1} + \varepsilon_T V_{T-1}) - (1 - \lambda)V_{T-1}[1 + \sum_{j=1}^{l_T-1} r_T^j x^*_T]
\]

Notice that
\[
\sum_{j=1}^{l_T-1} r_T^j x^*_T = \sum_{j=1}^{l_T-1} r_T^j \frac{\varepsilon_T}{q_T^j}
\]
\[
r_T^T x^*_T V_{T-1} = r_T^T [\left( \sum_{j=1}^{k_T} \frac{1}{q_T^j} \right)^{-1} \sum_{j=1}^{k_T} \frac{1}{q_T^j} - \varepsilon_T] \sum_{j=1}^{l_T-1} \frac{1}{q_T^j}
\]
\[
= r_T^T [\left( \sum_{j=1}^{k_T} \frac{1}{q_T^j} \right)^{-1} \sum_{j=1}^{k_T} \frac{1}{q_T^j} - \varepsilon_T] \sum_{j=1}^{l_T-1} \frac{r_T^j}{q_T^j}
\]
\[
= 1 + \sum_{j=1}^{l_T-1} r_T^j x^*_T + r_T^T x^*_T
\]
\[
= 1 + \sum_{j=1}^{l_T-1} r_T^j \frac{\varepsilon_T}{q_T^j} + \left( \sum_{j=1}^{k_T} \frac{1}{q_T^j} \right)^{-1} \sum_{j=1}^{l_T-1} \frac{r_T^j}{q_T^j}
\]
\[
= 1 + r_T^T + \varepsilon_T \sum_{j=1}^{l_T-1} \frac{r_T^j - r_T^T}{q_T^j}
\]

Then
\[
f_{T-1}(V_{T-1}, w^1_{T-1})
\]
\[
= \frac{\lambda}{T} (w^1_{T-1} + \varepsilon_T V_{T-1}) - (1 - \lambda)V_{T-1}[1 + r_T^T + \varepsilon_T \sum_{j=1}^{l_T-1} \frac{r_T^j - r_T^T}{q_T^j}]
\]
\[
= \frac{\lambda}{T} w^1_{T-1} + \sum_{j=1}^{l_T-1} r_T^j (\frac{\lambda}{T} - (1 - \lambda))[(1 + r_T^T + \varepsilon_T \sum_{j=1}^{l_T-1} \frac{r_T^j - r_T^T}{q_T^j}]]
\]
\[
= \frac{\lambda}{T} w^1_{T-1} - \alpha_T V_{T-1}
\]

where
\[
\alpha_T = \left[ -\frac{\lambda}{T} + (1 - \lambda)[(1 + r_T^T + \varepsilon_T \sum_{j=1}^{l_T-1} \frac{r_T^j - r_T^T}{q_T^j}]]
\]

We summarize the two cases as the following:
\[
f_{T-1}(V_{T-1}, w^1_{T-1}) = \frac{\lambda}{T} w^1_{T-1} - \beta_T V_{T-1}
\]

where
\[
\beta_T = \begin{cases} 
\epsilon_T & \left( \sum_{l=1}^{k_T} \frac{1}{q_T^l} \right)^{-1} \leq \varepsilon_T \\
\alpha_T & \left( \sum_{l=1}^{k_T} \frac{1}{q_T^l} \right)^{-1} > \varepsilon_T 
\end{cases}
\]
Then we consider period $T - 2$th to period $T - 1$th by assuming that $(V_{T - 2}, w^1_{T - 2})$ is known.

\[
f_{T - 2}(V_{T - 2}, w^1_{T - 2}) = \min_{(x_{T - 1}, z_{T - 1})} E[f_{T - 1}(V_{T - 1}, w^1_{T - 1}) | (V_{T - 2}, w^1_{T - 2})]
\]

\[
= \min_{(x_{T - 1}, z_{T - 1})} E\{[\frac{1}{T} w^1_{T - 1} + \beta_T V_{T - 1}] | (w_{T - 2}, V_{T - 2})\}
\]

\[
= \min_{(x_{T - 1}, z_{T - 1})} \{\frac{1}{T} (w^1_{T - 2} + z_{T - 2}) + \beta_T V_{T - 1} | V_{T - 2} + \sum_{j=1}^{n} r_{T - 1j} x_{T - 1j}\}
\]

We need to solve the following problem denoted by $P_{T - 1}$:

\[
(P_{T - 1}) \begin{cases} 
\min_{(x_{T - 1}, z_{T - 1})} \frac{1}{T} (w^1_{T - 2} + z_{T - 1}) - \beta_T (V_{T - 2} + \sum_{j=1}^{n} r_{T - 1j} x_{T - 1j}) \\
\text{s.t.} \quad q_{T - 1j} x_{T - 1j} \leq z_{T - 1}, \quad j = 1, \ldots, n \\
q_{T - 1j} x_{T - 1j} \leq \varepsilon_{T - 1}, \quad j = 1, \ldots, n \\
x_{T - 1j} \geq 0, \quad j = 1, \ldots, n \\
\sum_{j=1}^{n} x_{T - 1j} = V_{T - 2}
\end{cases}
\]

where $q_{T - 1j} = E[R_{T - 1j} - r_{T - 1j}]$, $j = 1, \ldots, n$.

The method solving this problem is similar to that of $P_T$. First, we employ the following auxiliary problem:

\[
(P'_{T - 1}) \begin{cases} 
\min_{(x_{T - 1}, z_{T - 1})} \frac{1}{T} (w^1_{T - 2} + z_{T - 1}) - \beta_T (V_{T - 2} + \sum_{j=1}^{n} r_{T - 1j} x_{T - 1j}) \\
\text{s.t.} \quad q_{T - 1j} x_{T - 1j} \leq z_{T - 1}, \quad j = 1, \ldots, n \\
x_{T - 1j} \geq 0, \quad j = 1, \ldots, n \\
\sum_{j=1}^{n} x_{T - 1j} = V_{T - 2}
\end{cases}
\]

**Theorem 3.3** The optimal solution to $P_{T - 1}$ denoted by $(x_{T - 1}^*, z_{T - 1}^*)$ is as follows:

1) If $z_{T - 1}^* \leq \varepsilon_{T - 1} V_{T - 1}$, then $(x_{T - 1}^*, z_{T - 1}^*)$ is also an optimal solution to $P_{T - 1}$;

2) If $z_{T - 1}^* > \varepsilon_{T - 1} V_{T - 1}$, then the optimal solution to $P_{T - 1}$ is as follows:

If there exists $n \geq l_{T - 1} > k_{T - 1}$, such that

\[
\frac{\sum_{j=1}^{l_{T - 1}} \frac{1}{q_{T - 1j}}}{\sum_{j=1}^{k_{T - 1}} \frac{1}{q_{T - 1j}}} > \frac{\varepsilon_{T - 1}}{z_{T - 1}^*} \geq \frac{\sum_{j=1}^{k_{T - 1}} \frac{1}{q_{T - 1j}}}{\sum_{j=1}^{l_{T - 1}} \frac{1}{q_{T - 1j}}}
\]

then the optimal solution to $P_T$ is as follows:

\[
\frac{x_{T - 1j}^*}{V_{T - 2}} = \frac{\varepsilon_{T - 1}}{q_{T - 1j}}, \quad j = 1, \ldots, l_{T - 1} - 1
\]
\[
\frac{x_{T-1}^{*}}{V_{T-2}} = \left( \sum_{j \in A_{T-1}(\lambda)} \frac{1}{q_{T-1j}} \right)^{-1} \sum_{j=1}^{k_{T-1}} \frac{1}{q_{T-1j}} - \varepsilon_{T-1} \sum_{j=1}^{l_{T-1}-1} \frac{1}{q_{T-1j}}
\]

\[
x_{T-1j}^{*} = 0, j = l_{T-1} + 1, \ldots, n
\]

By some deduction, we have:

If \( z_{T-1}^{*} \leq \varepsilon_{T-1}V_{T-1} \), then

\[
f_{T-2}(V_{T-2}, w_{T-2}^1) = \frac{\lambda}{T} w_{T-2}^1 - c_{T-1}V_{T-2}
\]

where

\[
c_{T-1} = \beta_T (1 + a_{T-1}b_{T-1}) - \frac{\lambda}{T} a_{T-1}, a_{T-1} = \left( \sum_{j=1}^{k_{T-1}} \frac{1}{q_{T-1j}} \right)^{-1}, b_{T-1} = \sum_{j=1}^{k_{T-1}} \frac{r_{T-1j}}{q_{T-1j}}
\]

If \( z_{T-1}^{*} > \varepsilon_{T-1}V_{T-1} \), then

\[
f_{T-2}(V_{T-2}, w_{T-2}^1) = \frac{\lambda}{T} w_{T-2}^1 - \alpha_{T-1}V_{T-2}
\]

where

\[
\alpha_{T-1} = -\frac{\lambda \varepsilon_{T-1}}{T} + \beta_T [1 + r_{T-1}u_{T-1} + \varepsilon_{T-1} \sum_{j=1}^{l_{T-1}-1} \frac{r_{T-1j} - r_{T-1u_{T-1}}}{q_{T-1j}}]
\]

We can summarize the above two cases into the following formula:

\[
f_{T-2}(V_{T-2}, w_{T-2}^1) = \frac{\lambda}{T} w_{T-2}^1 - \beta_{T-1}V_{T-2}
\]

where

\[
\beta_{T-1} = \begin{cases} 
 c_{T-1} \left( \sum_{l=1}^{k_{T-1}} \frac{1}{q_{T-1l}} \right)^{-1} \leq \varepsilon_{T-1} \\
 \alpha_{T-1} \left( \sum_{l=1}^{k_{T-1}} \frac{1}{q_{T-1l}} \right)^{-1} > \varepsilon_{T-1} 
\end{cases}
\]

(3.8)

Then for any \( t = 1, \ldots, T - 1 \), we assume that

\[
f_t(V_t, w_t^1) = \frac{\lambda}{T} w_t^1 - \beta_{t+1}V_t
\]

We will show

\[
f_{t-1}(V_{t-1}, w_{t-1}^1) = \frac{\lambda}{T} w_{t-1}^1 - \beta_tV_{t-1}
\]
where

\[
\beta_t = \begin{cases} 
\frac{c_t}{(\sum_{l=1}^{k_t} \frac{1}{q_{tl}})^{-1}} \leq \varepsilon_t \\
\frac{\alpha_t}{(\sum_{l=1}^{k_t} \frac{1}{q_{tl}})^{-1}} > \varepsilon_t 
\end{cases}
\]

where

\[
c_t = (1 - \lambda) \beta_{t+1} (1 + a_t b_t) - \frac{\lambda}{t} a_t, a_t = (\sum_{l=1}^{k_t} \frac{1}{q_{tl}})^{-1}, b_t = \sum_{l=1}^{k_t} \frac{r_{tl}}{q_{tl}}
\]

\[
\alpha_t = -\frac{\lambda \varepsilon_t}{t} + (1 - \lambda) \beta_{t+1} [(1 + r_{tl} + \varepsilon_t \frac{\sum_{j=1}^{n} r_{tj} - r_{tj}}{q_{tl}}]
\]

Consider the \( t \)th period assuming that \((V_{t-1}, w_{t-1}^1)\) is known, we have

\[
f_{t-1}(V_{t-1}, w_{t-1}^1) = \min_{(x_t, z_t)} E[f_t(V_t, w_t^1)|(V_{t-1}, w_{t-1}^1)] = \min_{(x_t, z_t)} \frac{1}{T} (w_{t-1}^1 + z_t) - \beta_t (V_{t-1} + \sum_{j=1}^{n} r_{tj} x_{tj})
\]

We need to get the optimal solution of the following problem:

\[
(P_t) \begin{cases} 
\min & \frac{1}{T} (w_{t-1}^1 + z_t) - \beta_{t+1} (V_{t-1} + \sum_{j=1}^{n} r_{tj} x_{tj}) \\
\text{s.t.} & q_{tj} x_{tj} \leq z_t, j = 1, \cdots, n \\
& q_{tj} x_{tj} \leq \varepsilon_t V_{t-1}, j = 1, \cdots, n \\
& x_{tj} \geq 0, j = 1, \cdots, n \\
& \sum_{j=1}^{n} x_{tj} = V_{t-1}
\end{cases}
\]

where \( q_{tj} = E[R_{tj} - r_{tj}], j = 1, \cdots, n \).

The method of solving \( P_t' \) is similar to that of \( P_T \) and \( P_{T-1} \). We do not repeat it again and give the result directly. Denote by

\[
(P_t') \begin{cases} 
\min & \frac{1}{T} (w_{t-1}^1 + z_t) - \beta_t (V_{t-1} + \sum_{j=1}^{n} r_{tj} x_{tj}) \\
\text{s.t.} & q_{tj} x_{tj} \leq z_t, j = 1, \cdots, n \\
& x_{tj} \geq 0, j = 1, \cdots, n \\
& \sum_{j=1}^{n} x_{tj} = V_{t-1}
\end{cases}
\]

Assume that the optimal solution to \( P_t' \) is \((x_t^*, z_t^*)\). And \( A_t(\lambda) = \{1, \cdots, k_t\} \).

**Theorem 3.4** The optimal solution to \( P_t \) is as follows:

1) If \( z_t^* \leq \varepsilon_t V_{t-1} \), then \((x_t^*, z_t^*)\) is also an optimal solution to \( P_t \);
2) If \( z^*_t > \varepsilon_t V_{t-1} \), then the optimal solution to \( P_t \) is as follows:

If there exists \( n \geq l_t > k_t \), such that

\[
\sum_{j=1}^{k_t} \frac{1}{q_{tj}} > \varepsilon_t V_{t-1} - \sum_{j=1}^{l_t-1} \frac{1}{q_{tj}} \]

then the optimal solution to \( P_t \) is as follows:

\[
x^*_{tj} = \frac{\varepsilon_t V_{t-1}}{q_{tj}}, \quad j = 1, \ldots, l_t - 1
\]

\[
x^*_{t_l} = z^*_t \sum_{j=1}^{k_t} \frac{1}{q_{tj}} - V_{t-1} \varepsilon_t - \sum_{j=1}^{l_t-1} \frac{1}{q_{tj}}
\]

\[
x^*_{tj} = 0, \quad j = l_t + 1, \ldots, n
\]

\[
z^*_{t} = \varepsilon_t V_{t-1}
\]

Substituting \((x^*_{t}, z^*_{t})\) to the \( f_{t-1} \), we can get the final result.

Based on the above result, we can get the expected total wealth at the end of the \( T \) periods:

\[
E(V_T) = (1 + \beta_T)(1 + \beta_{T-1}) \cdots (1 + \beta_1)V_0
\]

where

\[
\beta_t = \begin{cases} 
  c_t & (\sum_{l=1}^{k_t} \frac{1}{q_{lu}})^{-1} \leq \varepsilon_t \\
  \alpha_t & (\sum_{l=1}^{k_t} \frac{1}{q_{lu}})^{-1} > \varepsilon_t
\end{cases}
\]

where

\[
c_t = \beta_{t+1}(1 + a_t b_t) - \frac{\lambda}{t} a_t, \quad a_t = \left(\sum_{j=1}^{k_t} \frac{1}{q_{tj}}\right)^{-1} b_t = \sum_{j=1}^{k_t} \frac{r_{tj}}{q_{tj}}
\]

\[
\alpha_t = -\frac{\lambda \varepsilon_t}{t} + \beta_{t+1}[1 + r_{u_t} + \varepsilon_t \sum_{j=1}^{l_t-1} \frac{r_{tj} - r_{u_t}}{q_{tj}}]
\]

Finally, we give an example to demonstrate the adoption of the above model.

**Example.** We use the data from Markowitz (1959): returns on three securities are given in Table 1. An investor has one unit of wealth at the beginning of 1949.
Suppose that an investor was trying to find the best allocation of his/her wealth among three risky securities from 1949-1954 in order to maximize his/her terminal wealth for a given risk level.

Using the data in Table 1, we obtain the followings estimation: the expected return rate of $A.m.&T$, $G.M$ and $A.T.&S$ is 0.0654, 0.1734 and 0.1981 respectively. We denote

$$r_{t1} = 0.0654, r_{t2} = 0.1734, r_{t3} = 0.1981$$

Obviously, $r_{t1} < r_{t2} < r_{t3}$. We assume that $\lambda = 1/2$. $\varepsilon = (0.06, 0.07, 0.016, 0.1, 0.1, 0.05)$.

$$x_1 = (0.1357, 0.4066, 0.4576), \quad x_2 = (0.3324, 0.6226, 0.2150)$$

$$x_3 = (0.2101, 0.9964, 0.1358), \quad x_4 = (0.1778, 0.7185, 0.6596)$$

$$x_5 = (0.2704, 0.7426, 0.8097), \quad x_6 = (1.4286, 0.2338, 0.4673)$$

The expected wealth is $E(V) = 2.3562$. The real wealth is $V = 2.8057$. The risk is

$$w(x) = (0.06, 0.08190451, 0.021478198, 0.155586165, 0.182277126, 0.12659)$$
If we use simple period model to invest in this 3 assets and get its return. The optimal strategy is $x^* = (0.4105, 0.3320, 0.2575)$. The wealth is $V' = 2.2217$. The risk is $w'(x) = (0.0984, 0.0981, 0.0407, 0.0719, 0.0815, 0.1798)$

4. CONCLUSION AND REMARKS

In this paper, we present a new multiperiod portfolio model with maximum absolute deviation model. The investor is assumed to maximize the expected wealth at the end of the final period and minimize the risk which is defined as the average of the risk in all periods. At the same time, The risk in every period can not be above the given level. The closed form of the optimal strategy is given via dynamic programming method.

Noticing that in this paper, short selling is not allowed. Such an assumption is assumed to get the analytical solution. However, if we relax this restriction, i.e., short selling is permitted, can we can get the optimal strategy for the same problem? Since the wealth is assumed to be allocated completely in every period, if the investor puts some money aside for consumption or saving, only invest part of his wealth in every period, then how to solve such a practical problem? What is the optimal strategy if there is a riskless asset in the market? Moreover, after Konno et al give the MAD model in 1990, 15 years passed but there are little literature focus on the extending this model to multiperiod case. Such a work is meaningful, interesting but also challengeable and needs to further study.

APPENDIX

Obviously, Theorem 3.2, 3.3,3.4 can be solved by the same method. We first simplified the problems $P_T$, $P_{T-1}$, $P_t$ as the following problem:

$$
\begin{align*}
P_t \quad \left\{ \begin{array}{l}
\min \quad & \frac{\lambda}{T} y - (1 - \lambda) \sum_{j=1}^{n} r_j x_j \\
\text{s.t.} & \sum_{j=1}^{n} x_j = U \\
& a_j x_j \leq y, j = 1, \cdots, n; \\
& a_j x_j \leq \varepsilon U, j = 1, \cdots, n; \\
& x_j \geq 0, j = 1, \cdots, n;
\end{array} \right.
\end{align*}
$$
To solve this problem, we need to introduce the following auxiliary problem, which is the simplified form for $P'_T$, $P'_{T-1}$, $P'_I$:

$$P_{II} \begin{cases} \min & \frac{1}{T} y - (1 - \lambda) \sum_{j=1}^{n} r_j x_j \\ \text{s.t.} & \sum_{j=1}^{n} x_j = U \\ & a_j x_j \leq y, j = 1, \ldots, n; \\ & x_j \geq 0, j = 1, \ldots, n; \end{cases}$$

where $r_1 \leq r_2 \leq \cdots r_n \leq 0$, $a_j > 0$, $j = 1, \ldots, n$ and $U > 0$.

We can get the solution to $P_I$ by solving $P_{II}$. Noticing that the solution to $P_{II}$ is known (see Cai (2000)), we denote its solution as $(x^*, y^*)$.

**Lemma A.1** The solution to $P_I$ denoted by $(x^*, z^*)$, where $x^* = (x^*, \ldots, x^*)'$, is as follows:

$$x^*_j = \begin{cases} \frac{U}{a_j} \left( \sum_{i \in A} \frac{1}{a_i} \right)^{-1}, & j \in A \\ 0, & j \not\in A \end{cases}$$

$$y^* = U \left( \sum_{j \in A} \frac{1}{a_j} \right)^{-1}$$

where $A$ is determined by:

1. If there exists an integer $p \in [0, n - 2]$ such that

$$\frac{r_n - r_{n-1}}{a_n} < \frac{\lambda}{1 - \lambda}$$

and

$$\frac{r_n - r_{n-2}}{a_n} + \frac{r_{n-1} - r_{n-2}}{a_{n-1}} < \frac{\lambda}{1 - \lambda}$$

$$\frac{r_n - r_{n-p}}{a_n} + \frac{r_{n-1} - r_{n-p}}{a_{n-1}} + \cdots + \frac{r_{n-p+1} - r_{n-p}}{a_{n-p+1}} < \frac{\lambda}{1 - \lambda}$$

and

$$\frac{r_n - r_{n-p-1}}{a_n} + \frac{r_{n-1} - r_{n-p-1}}{a_{n-1}} + \cdots + \frac{r_{n-p} - r_{n-p-1}}{a_{n-p}} \geq \frac{\lambda}{1 - \lambda}$$

then

$$A = \{n, n - 1, \ldots, n - p\}.$$

2. Otherwise, if the above condition is not satisfied by any integer $p \in [0, n - 2]$, then

$$A = \{n, n - 1, \ldots, 1\}.$$
Obviously, $x_j^*$ satisfy that $a_j x_j^* = y^*$, $j \in A$, $a_j x_j^* = 0$ for $j \notin A$. Here, $A = \{n, n-1, \ldots, n-p\}$ or $\{n, n-1, \ldots, 1\}$. In the following discussion, for conveniently notation, we denote $A = \{1, \ldots, k\}$, $k \leq n$. It is worth noticing that the rank of assets in $A$ now is converse to the original one. That is for $i, j \in \{1, \ldots, k\}$, if $i < j$, then $r_i \geq r_j$.

Now we consider the solution to $P_I$. It is easy to know that if $y^* \leq \varepsilon U$, then $(x^*, y^*)$ is also the optimal solution to $P_I$. If $y^* > \varepsilon U$, then the feasible solution of $P_I$ may be zero if $\varepsilon \rightarrow 0$. This means that the investor is so conservative that do not want to bear any risk. In the following discussion, we focus on under what condition, problem $P_I$ has an optimal solution.

The optimal solution to $P_1$ can be known by the following steps:

step1 if there exists $l$ with $n > l > k$ such that

$$\sum_{j=1}^{k} \frac{1}{a_j} x_j^* > \frac{\varepsilon U}{y^*} \geq \sum_{j=1}^{l-1} \frac{1}{a_j} x_j^*$$

Step 2: The optimal solution to $P_I$ is:

$$x_j^{**} = \frac{\varepsilon U}{a_j}, \quad j = 1, \ldots, l - 1$$

$$x_l^{**} = y^* \sum_{j=1}^{k} \frac{1}{a_j} - \varepsilon U \sum_{j=1}^{l-1} \frac{1}{a_j}$$

$$x_j^{**} = 0, \quad j = l + 1, \ldots, n$$

$$y^{**} = \varepsilon U$$

**Theorem A.1.** The solution in step 2 is a feasible solution to $P_I$.

**Proof.** Since $j = 1, \ldots, k$, 

$$x_j^{**} = \frac{\varepsilon U}{a_j} = x_j^* - \frac{y^* - \varepsilon U}{a_j}$$

We have

$$\sum_{j=1}^{n} x_j^{**} = U - \sum_{j=1}^{k} \frac{y^* - \varepsilon U}{a_j} + \sum_{j=k+1}^{l-1} \frac{\varepsilon U}{a_j} + x_l^{**}$$

$$= U - y^* \sum_{j=1}^{k} \frac{1}{a_j} + \sum_{j=1}^{l-1} \frac{\varepsilon U}{a_j} + x_l^{**}$$

$$= U$$

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Obviously, we know that \( x_i^{**} \geq 0 \), then \( a_i x_i^{**} \leq \varepsilon U \). Hence, the solution in step 2 is a feasible solution to \( P_I \).

Following discussion will show that the solution in Step 1 and Step 2 is an optimal solution to \( P_I \). We separate the proof into theorems. First, we will show that if we cannot find \( l \) which satisfy (1) and (2), then the problem \( P_I \) has no feasible solution. Second, we will show that the solution in step 1 and step 2 is the optimal solution to \( P_I \).

**Theorem A.2** If we cannot find \( l \) satisfy Step 1, then the feasible solution to \( P_I \) is empty.

**Proof.** If there is no \( l \) satisfy Step 1, then

\[
y^* \sum_{j=1}^{k} \frac{1}{a_j} > \varepsilon U \sum_{j=1}^{n} \frac{1}{a_j}
\]

Noticing that \( a_j x_j^* = y^*, \ j = 1, \ldots, k \) and \( \sum_{j=1}^{k} x_j^* = y^* \sum_{j=1}^{k} \frac{1}{a_j} = U \).

If \( x_j \) is a feasible solution to \( P_I \), then \( a_j x_j \leq \varepsilon U, \ j = 1, \ldots, n \).

Noticing that \( U = \sum_{j=1}^{n} x_j \leq \varepsilon U \sum_{j=1}^{n} \frac{1}{a_j} \), we have

\[
y^* \sum_{j=1}^{k} \frac{1}{a_j} \leq \varepsilon U \sum_{j=1}^{n} \frac{1}{a_j}
\]

which is a contradiction. Hence, \( P_I \) has no feasible solution.

**Theorem A.3** Feasible solution in Step 1 and Step 2 is a optimal solution to \( P_I \).

**Proof.** For problem \( P_{II} \), we have

\[
\sum_{j=1}^{k} \frac{r_j - r_{k+1}}{a_j} \geq \frac{\lambda}{1 - \lambda}
\]

Considering that \( l > k, \ r_l \leq r_{k+1} \), we have

\[
\sum_{j=1}^{l} \frac{r_j - r_l}{a_j} > \sum_{j=1}^{k} \frac{r_j - r_l}{a_j} \geq \sum_{j=1}^{k} \frac{r_j - r_{k+1}}{a_j} \geq \frac{\lambda}{1 - \lambda}
\]
The lagrange function of problem $P_{II}$ is

$$L_f(x_j, y, \mu_j, p_j, s) = \frac{\lambda}{T} y -(1-\lambda) \sum_{j=1}^{n} r_j x_j + \sum_{j=1}^{n} \mu_j(r_j x_j - y) + \sum_{j=1}^{n} l_j(r_j x_j - \varepsilon U) - \sum_{j=1}^{n} p_j x_j + s(\sum_{j=1}^{n} x_j - U)$$

The Kuhn-Tucker condition is as follows:

$$\frac{\partial L}{\partial y} = \frac{\lambda}{T} - \sum_{j=1}^{n} \mu_j = 0, \quad \frac{\partial L}{\partial x_j} = -(1-\lambda)r_j + a_j \mu_j + l_j a_j + s - p_j = 0$$

$$\sum_{j=1}^{n} x_j = U, \quad \mu_j(a_j x_j - y) = 0, l_j(a_j x_j - \varepsilon U) = 0,$$

$$p_j x_j = 0, \quad \mu_j \geq 0, l_j \geq 0, p_j \geq 0, x_j \geq 0, j = 1, \cdots, n$$

Since $P_{II}$ is a linear program, the KT point is an optimal solution.

According to step 1 and step 2, we have $y^{**} = \varepsilon U$, $a_j x_j^{**} = \varepsilon U$, for $j = 1, \cdots, l - 1$ and $a_l x_l^{**} \leq \varepsilon U$, $x_j^{**} = 0$ for $j > l$ and $\sum_{j=1}^{n} x_j^{**} = U$.

It is easy to verify that the following Lagrange multiplier and KT point satisfy KT condition.

$$p_j = 0, \quad j = 1, \cdots, l$$

$$p_j = (1-\lambda)(r_l - r_j) \geq 0, \quad j = l + 1, \cdots, n$$

$$s = (1-\lambda)r_l$$

$$\mu_j = \frac{\lambda}{T} \frac{r_l - r_l - 1}{a_j}, \quad j = 1, \cdots, l - 1$$

$$\mu_j = 0, \quad j = l, \cdots, n$$

$$l_j = (1-\lambda)\frac{r_j - r_l}{a_j} - \frac{\lambda}{T} \frac{r_l - r_l - 1}{a_j}, \quad j = 1, \cdots, l - 1$$

$$l_j = 0, \quad j = l, \cdots, n$$

Then we complete the proof.

REFERENCES


