Risk-shifting Incentive Problem and Debt Concession

Masatoshi Miyake †, Hiroshi Inoue †

1 School of Management, Tokyo University of Science, Japan

Abstract—Risk-shifting incentive problem in financial contracting, which typically arises in the context of a lender-borrower relationship, stands for the borrower’s incentive to influence the risk of his project. Thus, the borrower can increase the value of his payoff at the expense of the lender. However, the lender also may have a risk incentive under some circumstances so that he may render it difficult to do efficient debt renegotiation. In this paper, we focus on not only borrower’s risk incentive but lender’s risk incentive problems and the relation between the both sides is discussed since the lender’s risk incentive may cause to accelerate the possibility of borrower’s risk incentive. Thus, when either side even selects a risky project instead of a safe one the inefficiencies of resource distribution are developed. We analyze the incentive problem, in which option pricing theory is used and some barriers must be attached to the option pricing formula in order for the formula to be justified.

Keywords—Barrier option, Debt concession, Lender and borrower, Option pricing, Risk-shifting incentive

1 Introduction

Moral hazard problem, which has been broadly studied in economics, financial engineering and other areas, is understood as one of inefficiency to distribute the resources. It is interpreted that after one person, who has much more information about the relevant area, finished his financial contract he may change his behavior and attitude toward his investment, causing some trouble to the other person. In general, it is not possible to supervise monitoring his behavior, and if it is even possible it costs much so that the moral hazard problem may be easily generated. When considering this moral hazard in financial contract, there may exist some asymmetric information between a principal (financial intermediary) as a lender and an agent as a borrower (company). Agent who has an advantage over principal may take some unperceived behavior or keep useful information secret, having some incentive to pursue preferentially his own profit.

Jensen and Meckling[2] propose risk-shifting problem as one typical incentive problem. They study incentive problems that, when an agent’s behavior cannot be perceived by a principal, the agent breaks their contract and invests in a risky project with higher return after the agent borrowed money from the principal. Such an incentive problem causes a loss to the principal who is at an inferior position, as a result it brings to social inefficiency. In a serious situation, the principal would not have a contract with the agent, causing financial activity and market to be withered. In order to solve the incentive problem above, it will be advised that the lender should give the borrower some stimulus that does not make the lender tell a lie and act undesired behavior when the lender and the borrower have a contract of principal-agent relation. In other words, the lender is required to lead normally the borrower who follows the path of honest behaviour. If such an appropriate incentive contract is designed the problem generated by asymmetric information is relieved so that efficient allocation of resources and social welfare are possibly improved. Thus, what incentive contract is suitable and how it is optimally designed become the key to solve the asymmetric information problem.

As a mean to mitigate risk incentive problems in firms Isagawa[3] and Isagawa & Yamashita[4] focus on strategy of a lender and discuss incentive problems between on both the lender and the borrower, and then they point out that debt concession of the lender is one of effective ways in the light of the efficiencies of resource distribution.

Thus, if the debt was exempted the risk-incentive problem of the borrower can be resolved, so that the borrower takes a safer project with an increase in values. Then, the total values of the borrower and the expected amount of money that the lender is able to collect may result in increase.

In this paper, we consider the risk-shifting problem in a framework of option pricing theory, in which a lending bank lends money to a borrowing firm for a finite time of period. We note that for the option pricing formula (Black-Scholes model) to be justified some barriers need to be attached to the formula since in the standard B-S equity valuation model shareholders are always supposed to select infinite-volatility projects. Also, in particular, we consider the situation in which the borrower has plural projects to invest instead of single one.

This paper is organized as follows. Section 2 describes the structure of contract between a lender and a borrower. In section 3, risk-shifting incentive problem is explained with option pricing theory. Section 4 talks about debt concession. Lender’s risk incentive with debt concession is discussed in section 5. Finally, conclusion is in section 6.

2 Structure of Contract

We consider a financial intermediary (a lender) and a firm (a borrower) that reach the following agreement. At initial time, a financial intermediary (a lender) lends money $D$ to a borrower in exchange for a promise by the lender to pay him $\bar{D} = D \exp(r(T-t))$ at maturity time $T$, and $r$ is interest rate for the borrower.

Then, the borrower invests its money after he received the money from the lender. The borrower selects an asset among many possible investing projects. At initial time the values of all assets are equivalent and the expected rate of returns is fixed with riskless interest rate $r$ but the risk $\sigma$ is different for all assets. Thus, at time $\tau(t \leq \tau \leq T)$ the borrower can select to invest all funds in any one of a series of project $i = 1 \cdots N$, whose value dynamics are given as geometric Brownian motion

$$dX_i = rX_i dt + \sigma_i X_i dW \quad (i = 1 \cdots N)$$

(1)

where $dW$ is a standard Wiener process. The borrower can, at any time, switch to another project with different risks after he invested. In order to solely examine the influence of risk-shifting incentive, assume all assets have the same maturity $T$ and the final return...
$X(\sigma)$ can be observed by both the lender and the borrower at maturity time $T$. If the value of the asset drops to around default barrier $L$, then the lender perceives the situation so that the project is immediately suspended and the contract between the lender and borrower is terminated. At that time, the possible collective loan by the lender is $L$ and the payoff by the borrower is 0. At maturity time $T$ the borrower pays $V_L(\sigma, \bar{D})$ back to the lender, and its payoff of the borrower is $X(\sigma) - V_L(\sigma, \bar{D})$. Also, the lender and the borrower have limited liability.

3 Risk-shifting Incentive Problem for Borrower and Lender

3.1 Financial contract and option theory

Stakeholders in financial contract are lender and borrower. In general, payoff structure of a borrower implies long position for call option for which asset of project is considered as underlying asset and a loan payment as strike price. On the other hand, a lender is in short position for the whole values of asset and call option. Therefore, since the downside risk for the borrower is limited and at the same time profit is not limited for upside, the borrower may have incentive to invest a project involving higher risk.

In applying option pricing theory for the payoff structure, as volatility in Black-Scholes formula gets increase the value of call option borrower possess increases. In contrast, while the lender cannot receive profit for upside when highly risky project is successful he also receives a loss for downside when the project fails. This implies that the lender who has short position for call option makes lender’s value increase by controlling volatility. But, some caution with respect to using Black-Scholes (B-S)formulae must be recognized in the light of volatility as mentioned in Introduction. In fact, Chesney and Gibson-Astner[10] point out that in the standard Black-Scholes equity valuation model, shareholders always select infinite-volatility projects. Smith and Warner[11] and Green[12] claim that standard option pricing modelling along B-S lines can be taken only as suggestive however, because it does not recognize that firm value is endogenous.

Therefore, in this study we even takes its dynamics of asset value of project before maturity into account. The reason for that is when asset value reaches at default boundary before maturity $T$ the lender may recognize the fact, and hence he can make the project immediately suspend. In general, the lender cannot supervise monitoring the details of the asset value, but it is not considered as the reality for a manager to leave the behaviour of the borrower to take it own course till the collective loan becomes 0. Thus, it is often observed in financial contract that the contract is forced to be suspended before maturity and the asset is withdrawn.

3.2 Asset value of project

Consider a financial intermediary (lending bank) that lends money to a borrower (borrowing firm) for investment in several projects that are available only to the borrowing firm. The shareholder’s objective, in general, does not necessarily coincide with the firm value maximization when the firm is levered. Thus, we assume that the shareholders’ main interest can be stated as follows. That is, the shareholders of a levered firm will act so as to maximize the value of their shares. The asset value of project $X_t$ is assumed to follow the usual geometric Brownian motion (1)

We introduce option theory in our framework, in which a barrier option involving knockout condition is considered. Note that the call price in the standard B-S model is a strictly increasing function of the underlying asset’s volatility. Thus, the B-S valuation framework does not give an accurate explanation of risk incentive as a function of a firm’s leverage. As mentioned in [10] shareholders are always supposed to select infinite-volatility project. Therefore, assuming that if the asset of project falls at default boundary the project is suspended, the project value $X(\sigma)$ at time $T$ can be expressed with barrier $L$ as follows.

$$X(\sigma) = \begin{cases} X_T & \text{min} (X_t > L) \\ L & \text{otherwise} \end{cases}$$

$$= \begin{cases} X_T & \text{min} (X_t > L) \\ 0 & \text{otherwise} \end{cases} + L - \begin{cases} 0 & \text{min} (X_t > L) \\ L & \text{otherwise} \end{cases}$$

$$= X_t e^{(T-t)\sigma^2} LN(d_1) - \left( \frac{L}{X_t} \right)^{\frac{\sigma^2}{2}} \frac{e^{(T-t)\sigma^2}}{\sqrt{2\pi}} LN(d_2) + L - \left( \frac{L}{X_t} \right)^{\frac{\sigma^2}{2}} \frac{e^{(T-t)\sigma^2}}{\sqrt{2\pi}} LN(d_2 - \sigma \sqrt{T-t})$$

$$\text{Value of } \Box$$

$$\text{Value of } \Box$$

where

$$d_1 = \frac{\ln(X_t/L) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}$$

$$d_2 = \frac{\ln(L/X_t) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}$$

Payoff structure of $\Box$ and $\Box$ consists of asset digital option and cash digital option involving knockout condition with a lower bound $L$. Hence, pricing for values of $\Box$ and $\Box$ is considered as follows.

Value of $\Box = C_1 - C_2$ and

$C_1$ value of $S_T$ when asset value falls over default boundary at maturity time $T$

$C_2$ value of $S_T$ when asset value even falls below the default boundary during option period and falls above the default boundary at maturity time $T$. 

$C_3$ value of $S_T$ when asset value falls over default boundary at maturity time $T$
Value of $D = C_2 - C_4$ and
$C_3$ value of $L$ when asset value falls over the default boundary at maturity time
$C_4$ value of $L$ when asset value even falls below the default boundary during option period and falls above the default boundary at maturity time.

**Remark 1:** There are some discussions how to formulize default boundaries. Longstaff and Schwartz [1995], Andou and Marusige [2001] think it as a constant which does not depend on time. On the other hand, Black and Cox [1976] and Briys and de Varenne [1997] assume increasing function adjusting with interest payment and with the ratio of collective loan at default time. In this study, we assume the default boundary does not depend on time and constant.

**Remark 2:** To avoid risk-shifting incentive problem any profit-sharing rule between the lender and the borrower can be characterized by a fixed payment and a certain number of put and call options [5, 7]. Actually, at any point in time, there exists an infinite number of profit-sharing rules avoiding risk-shifting though only one of these contracts is feasible.

### 3.3 Equity value of borrowing firm

Payoff of a borrower becomes 0 when the asset value falls below $L$ since default boundary is set. This payoff structure is knockout option. The pricing formula comes from Merton [18]. Denote the equity value of the borrowing firm at time $T$ by $V_B(\sigma, \bar{D})$ and the debt value by $\bar{D}$, respectively. The equity value $V_B(\sigma, \bar{D})$ can be expressed as follows,

$$V_B(\sigma, \bar{D}) = \begin{cases} \max \{X_T - \bar{D}, 0\} & \min \{X_T > L\} \\ 0 & \text{otherwise} \end{cases}$$

$$= X e^{(T-t)N(d_3)} - \bar{D}N(d_3 - \sigma \sqrt{T-t}) - X \left(\frac{X_T}{L}\right)^{\frac{2^4}{\sigma^4}} e^{(T-t)N(d_4)}$$

$$+ \left(\frac{X_T}{L}\right)^{\frac{2^4}{\sigma^4}} N(d_4 - \sigma \sqrt{T-t})$$

where

$$d_3 = \frac{\ln(X_T/\bar{D}) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \quad d_4 = \frac{\ln(L^2/X_T/\bar{D}) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}.$$  

Note that the equation of equity value of the borrowing firm still involves the barrier $L (< \bar{D})$. Then, as stated before, if the shareholders of a levered firm will act so as to maximize the value of their shares we look for an optimal volatility level. According to Chesney and Gibson-Asner the derivative of the equity value of the borrowing firm, $V_B(\sigma, \bar{D})$ with respect to $\sigma$ is obtained as

$$\frac{\partial V_B(\sigma, \bar{D})}{\partial \sigma} = X e^{(T-t)N(d_3)} \left\{ \exp \left(\frac{d_3^2}{2}\right) - \left(\frac{X_T}{L}\right)^{\frac{2^4}{\sigma^4}} \exp \left(\frac{d_4^2}{2}\right) \right\}$$

$$- \frac{4e^{(T-t)}}{\sigma^3} \ln \left(\frac{X_T}{L}\right) \left(\frac{X_T}{L}\right)^{\frac{2^4}{\sigma^4}} \left(X_T N(d_4) - \bar{D} \left(\frac{X_T}{L}\right)^{\frac{2^4}{\sigma^4}} \right) e^{-(T-t)N(d_4 - \sigma \sqrt{T-t})}$$

Thus, the optimal risk $\sigma^*_B$ which maximizes the borrower’s equity value is found as barrier price at

$$\frac{\partial V_B(\sigma, \bar{D})}{\partial \sigma} \bigg|_{\sigma=\sigma^*_B} = 0.$$ 

We solve equation (5) numerically because of the absence of a known closed form solution.

### 3.4 The possible collective bank loan for lender (Debt value)

The lender has a contract with the borrower to receive $\bar{D}$ at maturity time $T$. Also, if the asset value falls below $L$ before maturity time $T$ the lender suspends the project and collects loan money $L$. Therefore, the possible collective loan for the lender at time $T$, $V_L(\sigma, \bar{D})$ is obtained as the project value minus the equity value of the borrower, in other words,

$$V_L(\sigma, \bar{D}) = \begin{cases} \min \{X_T, \bar{D}\} & \min \{X_T > L\} \\ L & \text{otherwise} \end{cases}.$$ 

$$= X(\sigma) - V_B(\sigma, \bar{D})$$

The optimal risk $\sigma^*_L(< \sigma^*_B)$ that maximizes the possible collective bank loan is found when $\sigma^*_L \rightarrow 0$, and to find the approximate value of $\sigma^*_L$, for the convenience, set

$$V_L(\sigma, \bar{D}) \bigg|_{\sigma=\sigma^*_L} = \bar{D}.$$ 

In this case, we seek to find the values less than the maximum value of $\sigma$ so that the possible collective loan equals the maximum value of $\bar{D}$. In other words, we may find the value $\sigma$ so that the above relation holds.
3.5 Safe project and risky project

We assume two different types of projects which are a safe project $S$ and a risky project $R$, for which each has the optimal risk $\sigma^*_S$, $\sigma^*_R$, respectively. Generally speaking, under the assumption, a lender bank advances funds for a borrower, expecting that the borrower would choose the project $S$ to protect his profit. On the other hand, the borrower is likely to select the risky project $R$ in order to maximize his equity value contrary to the lender’s will.

$$\begin{pmatrix} \text{Project } S \\ \text{Risk : } \sigma_S = \sigma^*_S \end{pmatrix} \begin{pmatrix} \text{Project } R \\ \text{Risk : } \sigma_R = \sigma^*_R \end{pmatrix}$$

In particular, if the borrower is a levered firm the tendency to select the risky project $R$ becomes more and more eminent. Thus, the profit of the lender is hindered so that risk-shifting incentive problem comes into existence. We look at some example to see the relation of equity value and collective loan for different volatilities.

**Example1.** Let $X_t = 100, D = 100, r = 0.25, \sigma = 0.15, T = 3.5, L = 35$. Then, the following values for projects $S$ and $R$ are obtained from (5), (7) as $\sigma_S = 0.025$ and $\sigma_R = 1.7246$. Fig. 1 shows the transition of the equity value $V_B(\sigma, D)$ and the collective loan $V_L(\sigma, D)$ for different $\sigma$’s.

![Equity value and possible collective loan for volatility](image)

Figure 1 illustrates that as $\sigma$ increases to go beyond 0.025 the possible collective loan of the lender drops down and finally converges to the default boundary 35, and the stock value of the borrower increases with increase of $\sigma$, then it finally to remain to stay with the values around 156 after exceeding $\sigma = 1.725$.

Thus, if the borrower chooses the safe project $S$ in accordance with the lender’s intention, the lender can collect the whole loan. However, if the borrower chooses a risky project $R$ the collective loan of the lender drops from 169.046 to 39.621 and the most of the financed loan is run to waste, while the equity value of the borrower increases from 70.842 to 164.411.

<table>
<thead>
<tr>
<th>Table 1 Equity value and Possible collective loan for project $S, R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project $S$</td>
</tr>
<tr>
<td>Equity value</td>
</tr>
<tr>
<td>Possible Collective Loan</td>
</tr>
</tbody>
</table>

4 Debt concession

There are several ways to solve or improve the situation. Chesney and Gibson-Asner[10], Green[12] and Chiesa [14] discuss that debt with warrants or convertible debt can realign the incentives of bondholders and shareholders. Smith and Warner [11] and Kalay [13] mention about the role of the provisions of contracts. Gertner and Scharfstein [15] and other researchers point out that debt renegotiation often can’t be agreed on due to conflicts among multiple creditors when the borrowing firm has a number of different creditors.

One way may be waiver of an obligation by the financial institution. If a lending bank attaches the greatest importance to the equity value instead of the collective loan, then she does not agree to debt concession, leaving of the firm as it is. In this study, we consider waiver of an obligation by the financial institution.

Suppose that the borrower and the lender agree on debt concession, and hence the lender decreases debt loan from $D$ to $D^*$. In order to make debt concession, the new face value of the debt $D^*$ must satisfy some requirement. Isagawa[3], Isagawa and Yamashita[4] formalize agreement condition of lender and borrower on debt concession, which is expressed below. First, it is required that the act of debt concession leads the firm to undertake safe project $S$. In other words, under the new value of debt loan $D^*$, we may find the condition that the borrower would not change from the project $S$ to project $R$. That is,
This implies, under the new debt level, that the equity value when selecting project \( S \) is not lower than that when selecting project \( R \), which guarantees that risk incentive problem may not occur.

On the other hand, it is required that debt concession never decreases the value of the bank loan. In other words, it is assumed that the bank does not make a debt concession if such a concession would decrease the value of the loan. Thus, the condition that the lender agrees to waiver of an obligation is

\[
V_B(\sigma_R, \overline{D}) \leq V_B(\sigma_S, \overline{D}). 
\]

Similarly, the relation (9) shows that possible collective loan at the selection of \( S \) is not lower than that at the selection of \( R \), which also guarantees that risk incentive problem may not occur. (9) can be expressed as

\[
V_B(\sigma_S, D^\prime) \leq X(\sigma_R) - X(\sigma_S) + V_B(\sigma_R, \overline{D}),
\]

and this can be expressed along with (8)

\[
V_B(\sigma_R, D^\prime) \leq V_B(\sigma_S, D^\prime) \leq X(\sigma_S) - X(\sigma_R) + V_B(\sigma_R, \overline{D}).
\]

Figure 2 shows that the level of the debt value which satisfies the relation of (11).

The range of debt value that risk-shifting incentive by a borrower is not raised is found as \( 0 \leq D^*_B \leq 72.689 \), and the range for the lender to agree on waiver of an obligation becomes \( 39.621 \leq D^*_L \leq 100 \). Therefore, the values for satisfying the both conditions are obtained as \( 39.621 \leq D^*_L \leq 72.689 \). Thus, if the level of the debt value 100, before waiver of an obligation, decreases to 72.689, the corresponding equity value and collective loan after the waiver of an obligation become 167.199 for the equity value and 72.689 for the collective loan. In this situation, the risk-shifting incentive problem is dissolved so that the lender is able to recover to collect the loan whose value ranges 39.621 to 72.689.

| Table 2: Equity value and Possible collective loan after the waiver of an obligation |
|----------------------------------|------------------|-----------------|
| Project \( R \) | Project \( S \) after the waive of an obligation |
| Equity value | 164.411 | 167.199 |
| Collective loan | 39.621 | 72.689 |

5 Lender’s risk incentive with debt concession

In section 4 we found the conditions required for new debt loan, given that the lender agreed on debt concession. We discussed and examined the conditions that the lending bank would agree on debt concession. Let the face value of debt equity of the financial institution be \( B \) and the profit from the asset excluding financing to the borrowing firm \( A \). Denoting the difference of \( B \) and \( A \) by \( C=B-A \) the equity value and debt value of the lending bank be obtained as follows. Then, the equity value of the lending bank is affected only by the face value of risky loan. The equity value and the debt value of the lending bank can be figured out, which really depend on the value of \( C \).

5.1 Equity value of financial intermediary

Equity value of financial intermediary \( V_{L,E}(\sigma, \overline{D}) \) can be expressed as call option with barrier, where collective loan (3) from the borrower corresponds to underlying asset with strike price \( C \).

\[
V_{L,E}(\sigma, \overline{D}) = \max\left[ V_L(\sigma, \overline{D}) - C \right]
\]

\[
= \begin{cases} 
\min\left[X_T - \overline{D}\right] - \min\left[X_T - \overline{D}, C\right] & \text{min}_{t \leq s \leq T} \left(X_T > L\right) \\
\max\left[L - C, 0\right] & \text{otherwise}
\end{cases}
\]

\[ (12) \]
We examine the equity value \((12)\) separately for two cases (i) \(\overline{D} \leq C\), (ii) \(C < \overline{D}\).

(i) For \(\overline{D} \leq C\), since \(L < \overline{D}\), \(L < \overline{D} \leq C\) holds and \((12)\) becomes below.

\[
V_{L,E}(\sigma, \overline{D}) = \begin{cases} 
\min[X_T, \overline{D}] - \min[X_T, \overline{D}] \min_{t \in S^T} (X_T > L) & \text{if } \min_{t \in S^T} (X_T > L) \\
0 & \text{otherwise}
\end{cases}
\]

\[= 0 \] (13)

Namely, when asset state becomes worse \((\overline{D} \leq C)\) the possible collective loan from borrowing firm corresponds to debt value so that the equity value of financial institution becomes always 0 and does not depend on the risk of asset, not having own risk incentive.

(ii) For \(C < \overline{D}\), \(\min[X_T, \overline{D}, C]\) of \((12)\) becomes below, keeping \(C\)

\[
V_{L,E}(\sigma, \overline{D}) = \begin{cases} 
\min[X_T, \overline{D}] - \min[X_T, C] \min_{t \in S^T} (X_T > L) & \text{if } \min_{t \in S^T} (X_T > L) \\
\max[L - C, 0] & \text{otherwise}
\end{cases}
\]

Further, \((14)\) can be examined in three cases (a) \(C < 0\), (b) \(0 \leq C < L < \overline{D}\), (c) \(0 \leq L < C < \overline{D}\).

(a) For \(C < 0\), \((14)\) becomes below.

\[
V_{L,E}(\sigma, \overline{D}) = \begin{cases} 
\min[X_T, \overline{D}] - C \min_{t \in S^T} (X_T > L) & \text{if } \min_{t \in S^T} (X_T > L) \\
\max[L - C, 0] & \text{otherwise}
\end{cases}
\]

\[
= V_L(\sigma, \overline{D}) - C \] (15)

The optimal risk \(\sigma_{L,E}^*\) which maximizes equity value of \((15)\) is

\[
V_L(\sigma, \overline{D})\bigg|_{\max_{\sigma \geq \sigma_{L,E}^*}} = \overline{D} \] (16)

Therefore, when the asset state is healthy \((C < 0)\) the possible collective loan from borrowing firm belongs to equity value of financial institution, and some incentive for which collective loan may not decrease comes out, and then the optimal risk of equity value becomes equal to \((7)\) so that risk-shifting incentive problem comes into existence for borrower.

(b) For \(0 \leq C < L < \overline{D}\), \((14)\) becomes below.

\[
V_{L,E}(\sigma, \overline{D}) = \begin{cases} 
\min[X_T, \overline{D}] - \min[X_T, C] \min_{t \in S^T} (X_T > L) & \text{if } \min_{t \in S^T} (X_T > L) \\
\min[L - C, \max[L - C, 0]] & \text{otherwise}
\end{cases}
\]

In this case, the second term is \(C\) from payoff structure and hence

\[
V_{L,E}(\sigma, \overline{D}) = V_L(\sigma, \overline{D}) - C \] (17)

Project risk that equity value of \((17)\) is to be maximized is

\[
V_L(\sigma, \overline{D})\bigg|_{\max_{\sigma \geq \sigma_{L,E}^*}} = \overline{D} \] (18)

Hence, in the case of \(0 \leq C < L < \overline{D}\) the collective loan from borrower belongs to equity value of financial institution some incentive which may not decrease the collective loan appears as in \(C < 0\). Then, the optimal risk of equity value of financial institution becomes equal to \((7)\), raising risk incentive problem for borrower.

(c) For \(0 \leq L < C < \overline{D}\), \((14)\) becomes below

\[
V_{L,E}(\sigma, \overline{D}) = \begin{cases} 
\min[X_T, \overline{D}] - \min[X_T, C] \min_{t \in S^T} (X_T > L) & \text{if } \min_{t \in S^T} (X_T > L) \\
0 & \text{otherwise}
\end{cases}
\]

\[
= 0 \] (19)

Project risk that equity value of \((19)\) is to be maximized is

\[
(-V_B(\sigma, \overline{D}) + V_B(\sigma, C))\bigg|_{\max_{\sigma \geq \sigma_{L,E}^*}} = \overline{D} - C \] (20)

Hence, for \(0 \leq L < C < \overline{D}\) risk incentive may exist so that risk-shifting incentive problem occurs between the bank and the borrower.

5.2 Debt Value of Financial Institution
Debt value of financial institution, $V_{L,D}(\sigma, \bar{D})$, is expressed as the value that collective loan from the relevant borrower firm plus profit from asset not including loan to other firms $A$ and subtract equity value of the financial institution.

$$
V_{L,D}(\sigma, \bar{D}) = \min[V_L(\sigma, \bar{D}) + A, B]
= V_L(\sigma, \bar{D}) + A - V_{L,E}(\sigma, \bar{D})
$$

(21)

We examine debt value of (21) for two cases, (i) $\bar{D} \leq C$, (ii) $C < \bar{D}$.

(i) For $\bar{D} \leq C$ since the equity value becomes 0 and (21) becomes below.

$$
V_{L,D}(\sigma, \bar{D}) = V_L(\sigma, \bar{D}) + A
$$

(22)

Project’s risk which maximize debt value of (22), $\sigma^*_L,D$ is

$$
V_L(\sigma, \bar{D})|_{\max \sigma \geq \sigma^*_L,D} = \bar{D}
$$

(23)

In other words, when asset situation becomes unhealthy ($\bar{D} \leq C$), since collective loan from the relevant borrower firm goes to debt value of the financial institution some incentive which does not make the collective loan lose comes up. Hence, optimal risk of debt value becomes equal to (7) and risk-shifting incentive problem for borrower occurs.

(ii) In case of $C < \bar{D}$, three different cases (a) $C < 0$, (b) $0 \leq C < L < \bar{D}$, (c) $0 \leq L < C < \bar{D}$ are considered.

(a) For $C < 0$, equity value of financial institution becomes $V_L(\sigma, \bar{D}) - C$ by (15) so that (21) becomes the following.

$$
V_{L,D}(\sigma, \bar{D}) = V_L(\sigma, \bar{D}) + A - V_{L,E}(\sigma, \bar{D}) - C
= B
$$

(24)

Since, when asset state is healthy ($C < 0$) the debt $B$ can be paid the liabilities in full by profit $A$ which comes from other asset not including the loan to the relevant borrowing firm. The debt value of financial institution becomes always $B$ and does not depend on the risk of asset, and hence does not have own risk incentive.

(b) For $0 \leq C < L < \bar{D}$ equity value of financial institution is $V_L(\sigma, \bar{D}) - C$ by (17) (21) is expressed as below.

$$
V_{L,D}(\sigma, \bar{D}) = V_L(\sigma, \bar{D}) + A - V_{L,E}(\sigma, \bar{D}) + C
= B
$$

(25)

Therefore, for $0 \leq C < L < \bar{D}$ debt value of financial institution is always $B$ which is same as the situation of $C < 0$, does not depend on asset risk and does not have own risk incentive.

(c) For $0 \leq L < C < \bar{D}$ equity value of financial institution becomes $-V_B(\sigma, \bar{D}) + V_B(\sigma, C)$ by (19) and (23) becomes below.

$$
V_{L,D}(\sigma, \bar{D}) = V_L(\sigma, \bar{D}) + A + V_B(\sigma, \bar{D}) - V_B(\sigma, C)
= X(\sigma) - V_B(\sigma, C) + A
= V_L(\sigma, C) + A
$$

(26)

Project’s risk which maximize debt value of (26)

$$
V_L(\sigma, C)|_{\max \sigma \geq \sigma^*_L,D} = C
$$

(27)

Hence, for $0 \leq L < C < \bar{D}$ there exist some risk incentive with debt value of financial institution so that risk-shifting incentive problem occurs for lender. Also, the optimal risk is the numerical value replacing $\bar{D}$ of (7) by $C$.

Summarizing equity value and debt value for borrower and lender, the following table is obtained.

<table>
<thead>
<tr>
<th>Table 3 Debt value for borrower and lender</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C &lt; 0$</td>
</tr>
<tr>
<td><strong>Borrower</strong></td>
</tr>
<tr>
<td><strong>Lender</strong></td>
</tr>
<tr>
<td><strong>Debt value</strong></td>
</tr>
<tr>
<td><strong>Total</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\bar{D} \leq C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_B(\sigma, \bar{D})$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$C &lt; L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_L(\sigma, \bar{D})$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L &lt; C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
</tr>
<tr>
<td>$V_L(\sigma, \bar{D}) + A$</td>
</tr>
</tbody>
</table>
On the other hand, each optimal risk becomes below.

<table>
<thead>
<tr>
<th>Table 4 Optimal risk for borrower and lender</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Borrower</strong></td>
</tr>
<tr>
<td><strong>Lender</strong></td>
</tr>
<tr>
<td><strong>Debt value</strong></td>
</tr>
</tbody>
</table>

- When the asset state is healthy \( (C < 0) \) the debt value of financial intermediary is always \( B \). This means that the liabilities \( B \) are paid in full by profit \( A \) from assets excluding a loan to the relevant borrowing firm. Hence, the financial intermediary does not show any interest in risk incentive of the borrowing firm. On the other hand, the optimal risk of equity value of financial intermediary is equal to (7) (an incentive to expect project \( S \) occurs), and there may appear risk-shifting incentive problem for the borrowing firm. This implies that since the whole collective loan from the borrowing firm totally depends on the equity value, to increase collective loan is equivalent to increasing the equity value. Therefore, as similarly before, if the level of liability is reduced to 72.689 risk-shifting incentive problem is solved and the equity value of the lending bank does not remarkably drop. Thus, the financial intermediary would agree to debt concession in a position to protect equity value.

- When the asset state grows more serious \( (\overline{D} \leq C) \) the equity value of the lending bank becomes always 0. This shows that even though the whole promised amount of money \( \overline{D} \) from the borrowing firm was possibly collected the asset status is getting worse and in debt excess status, so that the whole collective loan goes to the liabilities. Therefore, he does not show any interest of risk incentive of borrowing firm. On the other hand, the optimal risk of debt value of the lending firm is equal to (7), and risk-shifting incentive problem for borrowing firm. The collective loan of borrowing firm belongs entirely to debt value. Therefore, as similarly before, if the level of liability is reduced to 72.689 risk-shifting incentive problem is solved and the value of liabilities of the lending bank does not go down. Thus, the financial intermediary would agree to debt concession in a position to protect debt value.

- When asset state is \( 0 \leq C < L < \overline{D} \) the debt value of financial institution is always \( B \). This is understood that the liabilities are not paid in full by profit from assets excluding a loan to the relevant firm but it is paid in full by collective loan \( L \) when reaching the default boundary. Hence, it does not show any interest for risk incentive of borrowing firm. On the other hand, the optimal risk of equity value of financial intermediary is equal to (7) and there may produce risk-shifting incentive problem for the borrowing firm. This implies that since the whole collective loan from the borrowing firm totally belongs to the equity value. Therefore, as similarly before, if the level of liability is reduced to 72.689 risk-shifting incentive problem is solved and the equity value of lending bank does not remarkably drop. Thus, financial intermediary would agree to debt concession in a position to protect equity value.

- When the asset details is \( 0 < L \leq C < \overline{D} \). In other words, in the case that is not possible to pay in full by profits from assets excluding a loan to borrowing firm and not even possible to pay in full with the collective loan \( L \) in the case of reaching the default boundary, we explain with Example 2 below.

**Example 2.** Let \( X_f = 100, D = 100, r = 0.25, r^r = 0.15, T = 3.5, L = 35, B = 170, A = 100 \). Then, the optimal risk for equity value and debt value of a lender which calculated by (20) and (27) are, respectively, \( \sigma^*_L,E \leq 0.024, \sigma^*_L,D \leq 0.085 \). Therefore, the optimal risk of lender which satisfies the both conditions becomes \( \sigma^*_L = 0.024 \). Assuming projects \( S, R \) as before, we obtain

\[
\begin{pmatrix}
\text{Project } S \\
\text{Risk : } \sigma_S = 0.024
\end{pmatrix}
\begin{pmatrix}
\text{Project } R \\
\text{Risk : } \sigma_R = 1.725
\end{pmatrix}
\]

Also, \( V_B(\sigma, D), V_{L,E}(\sigma, D), V_{L,D}(\sigma, D) \) for \( \sigma \) move as follows.
When risk $\sigma$ increases to go beyond 0.024 of equity value and beyond 0.085 of debt value of financial institution the collective possible loan has tendency to go down. Finally, the equity value becomes 0 and the debt value becomes $L+A=135$, which is obtained by adding profit from assets excluding financing loan to borrowing firm to the default boundary. If the borrower selects project $S$ to suit lender’s convenience the equity value of lender becomes $0.04699$ and debt value, $170$. On the other hand, if the borrower selects project with high risk $R$ against lender’s intention the equity value of lender decreases to $3.294$ and debt value $136.371$.

\[
\begin{array}{l|l|l}
\text{Project} & \text{Equity value} & \text{Debt value} \\
\hline
\text{Borrower} & 70.842 & 164.411 \\
\text{Lender} & 99.046 & 3.249 \\
\hline
\end{array}
\]

The condition that risk-shifting incentive problem is dissolved by debt concession and equity value and debt value of lender does not decrease is that when borrower selects project $S$ under the level of debt $D^*$ the equity value of borrower does not below the equity value when selects project $R$.

\[V_B(\sigma_R, D^*) \leq V_B(\sigma_S, D^*) \tag{28}\]

On the other hand, the condition that when lender selects project $S$ after debt concession the equity value of financial institution does not low the equity value when selects project $R$ before debt concession may be

\[V_{L,E}(\sigma_R, D) \leq V_{L,E}(\sigma_S, D^*) \tag{29}\]

Note that since debt value of lender does not influence by decreasing borrower’s debt it is ignored. Next, with (19) through (29) the following expression is obtained

\[-V_B(\sigma_R, D) + V_B(\sigma_R, C) \leq -V_B(\sigma_S, D^*) + V_B(\sigma_S, C) \tag{30}\]

Then, combining (28) and (30) we get

\[V_B(\sigma_R, D^*) \leq V_B(\sigma_S, D^*) \leq V_B(\sigma_S, C) + V_B(\sigma_R, D) - V_B(\sigma_R, C) \tag{31}\]

Applying parameter values the following is obtained
Debt range which does not raise risk-shifting incentive problem becomes $0 \leq \text{D}^1 \leq 72.689$. On the other hand, the range of debt which lender agrees with debt concession $73.249 \leq \text{D}^2 \leq 100$. Hence, the debt level satisfying (28) and (29) does not exist. Thus, when asset content is $0 < L < C < B$, financial institution does not agree to debt concession so that risk-shifting incentive problem between the both sides can not be solved, leading to ineffective investment.

6 Conclusion

In this paper, we discussed not only borrower’s incentive but lender’s incentive and its relation. First, it must be understood that we use a down side knock-out barrier option for equity values, so that the optimal volatility level of projects becomes finite. Thus, the standard B-S evaluation model is not sufficient in the light of the situation in which volatility is sensible for asset values and shareholders always select infinite-volatility projects. If the borrower chooses the safe project instead of the risky one the possible collective loan of the lender increases but the opposite situation is recognized when the borrower chooses the risky project. Also, we discussed and examined the conditions that the lending bank would agree on debt concession so as not to raise risk incentive by the borrower, in particular, when the borrower is a levered firm or at least unhealthy asset state. This implies lender’s attitude toward a borrower seems to be important and difficult since the lender is often required to show helpful effort from a moral point of view. We showed the relation between the borrower and lender with numerical examples under different circumstances.

Debt concession is certainly one effective way to avoid the risk-shifting incentive problem, and another important problem remains as the future work. That is described to further study some conditions that the lender may agree on the debt concession and need to refer to other possibilities like warrants or convertible debt.

References