Tail Decay Rates in Double QBD Processes and Related Reflected Random Walks

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A double quasi-birth-and-death (QBD) process is the QBD process whose background process is a homogeneous birth-and-death process, which is a synonym of a skip free random walk in the two dimensional positive quadrant with homogeneous reflecting transitions at each boundary face. It is also a special case of a 0-partially homogenous chain introduced by Borovkov and Mogul'skii [5]. Our main interest is in the tail decay behavior of the stationary distribution of the double QBD process in the coordinate directions and that of its marginal distributions. In particular, our problem is to get their rough and exact asymptotics from primitive modeling data. We first solve this problem using the matrix analytic method. We then revisit the problem for the 0-partially homogenous chain, refining existing results. We exemplify the decay rates for Jackson networks and their modifications.

Key words: quasi birth and death process; partially homogeneous chain; two dimensional queues; stationary distribution; rough decay rate; exact asymptotics; reflected random walk; Jackson network with server cooperation

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1. Introduction. This paper considers asymptotic behavior for a skip free random walk in the two dimensional positive quadrant of the lattice with homogeneous reflecting transitions at each boundary face. Take one of its coordinates as a level and the other coordinate as a phase, which is also called a background state. Then, this random walk can be considered as a continuous-time quasi-and-birth-and-death process, a QBD process in short, with infinitely many phases through uniformization due to the homogeneous transition structure. Since the level and phase are exchangeable, we call this reflected random walk a double QBD process. This process is rather simple, but has flexibility to accommodate a wide range of queueing models, including two node networks. It is also amenable to analysis by matrix analytic methods (e.g., see [19, 28]).

In general, a two dimensional reflected random walk is hard to study. For example, its stationary distribution is unknown except for special cases. Not only because of this but also for its own importance, researchers have studied the tail asymptotics of its stationary distribution. We are interested in rough and exact asymptotics. Here, the asymptotic decay is said to be exact with respect to a given function when the ratio of the probability mass function to the given function converges to unity as the level goes to infinity while it is said to be rough when the ratio of their logarithms converges to unity, which is typically used in large deviations theory.

Similar decay rate problems have been studied for a much more general two dimensional reflected random walk, so called, N-partially homogeneous chain in [5], where the N is a positive real number and specifies the depth of the boundary faces. Both the rough and exact asymptotics have been considered in all directions for the 0-partially homogeneous chain, which includes the double QBD process as a special case. However, the decay rate is not made explicit even for this special case, in particular, for the exact asymptotics, and there is no answer for the marginal distributions (see also [17]).

In this paper, we attack these problems in a different way from the large deviations approach which has been extensively used in the literature including [5, 17]. We are only concerned with the double QBD process, i.e., a skip free 0-partially homogeneous chain, and consider asymptotic behavior of its stationary distribution in the directions of coordinates. This allows us to use the nice geometric structure of the stationary distribution, which is called a matrix geometric form in the matrix analytic literature. Thus, we directly consider the stationary distribution itself.

We will find full spectra of a rate matrix for the matrix geometric form, and characterize the rough decay rates as solutions of an optimization problem. We then get the rough and exact asymptotics with help of graphical interpretation of the spectra, which can be done using only primitive modeling data, i.e., one step transition probabilities. This approach not only sharpens existing results under weaker assumptions for the decay rate problem on the double QBD process, but also enables us to find the rough and exact asymptotics of the marginal distributions. Since the decay rates are computed without using the stability conditions, it may be questioned whether the stability is guaranteed if the geometric decay rate is less than 1. We show that this is the case if some minor information is available.

To make clear our contributions, we revisit the decay rate problem under the framework of the 0-partially homogeneous chain. In this case, the rough decay rate in an arbitrary direction is considered. We present the results of [5] explaining what conditions are additionally required. Here, no skip free condition is needed, but the decay rates are less explicit due to this generality. Namely, it requires asymptotics of certain local processes, which are Markov additive processes obtained by removing the boundaries except for one of them, and the optimization problem that determines the decay rates is not solved there. Assuming a slightly stronger moment condition, we solve the optimization problem, and show that the asymptotics is obtained through the convergence parameters of the operator moment generating functions, called the Feynman-Kac transform, of the Markov additive kernels of the local processes. In this way, the results of [5] are refined, and compared with those for the double QBD process that are obtained in the first part.

We finally exemplify our computations of the decay rates using a modified Jackson network in which one server may help the other server. This problem has been considered in [9], but has not yet fully solved. We will solve this problem, and see how the boundary behavior affects the decay rates.

In the context of large deviations theory, the decay rate problem has been fairly well studied. For example, the so called sample path large deviations have been established for the stationary distribution of a multidimensional reflected process in continuous time generated by a reflection map (e.g., see [22]). This derives the rough decay rate in principle. However, one needs to identify a so called local rate function and to solve a variational problem. Both are generally hard problems, and individual effort has been dedicated to each specific model. In particular, Shwartz and Weiss [33] compute the rate functions for various models, and reduce the variational problem to a convex optimization problem. The approach of [5] is also based on large deviations theory, but takes one more step toward an explicit solution, concentrating on a two dimensional reflected random walk but allowing more complex boundary behavior. We make a further step in Section 5.2, which reproduces the results in [17].

A key issue to solve the decay rate problem in two dimensions is how to incorporate boundary effects to get a correct solution. The large deviations approach reduces this problem to find an optimal path that minimizes a cost along it. Instead of doing so, we simultaneously consider effects from the two boundary faces to get another optimization problem. This corresponds with how to choose the optimal path in the large deviations approach, but it is a new feature for the existing analytic approach. For example, the recent studies [26, 14, 16, 31] on the discrete-time QBD and $GI/G/1$-type processes with infinitely many phases assume stronger conditions for handling the boundary effect. A similar problem occurs in different analytic approach of Foley and McDonald [7, 8, 9].

Other than those two approaches, there is the much literature on the decay rate problem in two dimensions. For example, much effort has been dedicated to numerical approximations for the stationary distributions (e.g., see [1, 20]). In general, increasing boundaries makes the problem harder. As a first step, asymptotic behavior has been considered for the local processes that we have discussed. For example, Ignatiouk-Robert [18] shows how the convergence parameter of the Feynman-Kac transform determines the local rate function. There is some effort for computing the convergence parameter by the matrix analytic method (see, e.g., [14, 27]). The decay rate problem itself has been considered for various models with multiple queues, e.g., join shortest queues and generalized Jackson networks (e.g., see the literature in [7, 11, 12, 16, 34]).

This paper is made up of seven sections. In Section 2, we introduce the double QBD, and consider its basic properties. In Section 3, we prepare key tools for computing the decay rates. In Section 4, our main results are obtained, which gives the rough and exact asymptotics. In Section 5, we refine the results in [5], and compare them with our results. In Section 6, we exemplify the decay rates for Jackson network and its modifications. In Section 7, we remark that our main results on the rough decay rate are still valid under suitable changes of notation when the irreducibility condition which we have used is not satisfied. We also remark other possible extensions.

2. Double QBD process. Let $\{\boldsymbol{L}_n\}$ be a two dimensional Markov chain taking values in $S_+ \equiv \mathbb{Z}_+ \times \mathbb{Z}_+$, where $\mathbb{Z}_+ = \{0, 1, \ldots\}$, with transitions as depicted in Figure 1 below.

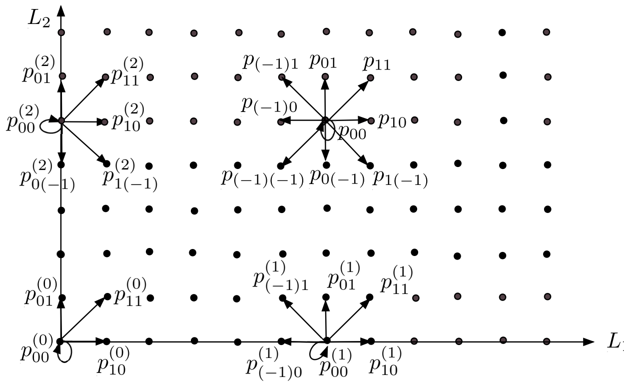

Figure 1: State transitions for the double QBD process

Thus, $\{\boldsymbol{L}_n\}$ is a skip free random walk in all directions, and reflected at the boundary $\partial S_+ \equiv \{(j,k) \in S_+; j = 0 \text{ or } k = 0\}$. To be precise, subdivide the boundary ∂S_+ into $\partial S_+^{(0)} \equiv \{(0,0)\}$ and $\partial S_+^{(i)} \equiv \{(j_1,j_2) \in \partial S_+ \setminus \partial S_+^{(0)}; j_{3-i} = 0\}$ for $i = 1,2$. Let $\{p^{(0)}_{jk}; j,k = 0,1\}$, $\{p^{(1)}_{jk}; j = 0,\pm1, k = 0,1\}$, $\{p^{(2)}_{jk}; j = 0,1, k = 0,\pm1\}$ and $\{p_{jk}; j,k = 0,\pm1\}$ be probability distributions on the sets of specified states (j,k), respectively. Denote random vectors subject to these distributions by $\boldsymbol{X}^{(0)} \equiv (X^{(0)}_1, X^{(0)}_2)$, $\boldsymbol{X}^{(1)} \equiv (X^{(1)}_1, X^{(1)}_2)$, $\boldsymbol{X}^{(2)} \equiv (X^{(2)}_1, X^{(2)}_2)$ and $\boldsymbol{X} \equiv (X_1, X_2)$, respectively. Then, $\{\boldsymbol{L}_n\} \equiv \{(L_1, L_2)\}$ is a Markov chain that has the transition probabilities:

$$P(\boldsymbol{L}_{n+1} = \boldsymbol{k} | \boldsymbol{L}_n = \boldsymbol{j}) = \begin{cases} P(\boldsymbol{X} = \boldsymbol{k} - \boldsymbol{j}), & \boldsymbol{j} > \boldsymbol{0}, \boldsymbol{k} \geq \boldsymbol{0}, \\ P(\boldsymbol{X}^{(i)} = \boldsymbol{k} - \boldsymbol{j}), & \boldsymbol{j} \in \partial S_+^{(i)}, \boldsymbol{k} \geq \boldsymbol{0}, i = 0,1,2, \\ 0, & \text{otherwise,} \end{cases}$$

where inequalities for vectors stand for their entry-wise inequalities.

We view $\{\boldsymbol{L}_n\}$ as a discrete time QBD process, which is a Markov chain that has two components, called a level and a phase. The level is nonnegative integer valued, but the phase is not necessarily so. The QBD process is usually delved in continuous time, but it can be formulated as a discrete time process by uniformization if its transition rates are bounded, which is the case here. For $\boldsymbol{L}_n \equiv (L_{1n}, L_{2n})$, we can take either one of L_{1n} and L_{2n} to be the level, so we refer to $\{\boldsymbol{L}_n\}$ as a double QBD process. We assume that this process is stable, that is,

(i) $\{\boldsymbol{L}_n\}$ is irreducible, aperiodic and positive recurrent.

Let $m_i = E(X_i)$ and $m^{(j)}_i = E(X^{(j)}_i)$ for $i,j = 1,2$. Then, Fayolle, Malyshev and Menshikov [6] characterize this stability in the following way.

Proposition 2.1 (Theorem 3.3.1 of [6]) *Assume that the double QBD process is irreducible and aperiodic, and either one of m_1 or m_2 does not vanish. Then, it is positive recurrent if and only if one of the following conditions holds.*

(S1) $m_1 < 0$, $m_2 < 0$, $m_1 m^{(1)}_2 - m_2 m^{(1)}_1 < 0$ *and* $m_2 m^{(2)}_1 - m_1 m^{(2)}_2 < 0$.

(S2) $m_1 \geq 0$, $m_2 < 0$ *and* $m_1 m^{(1)}_2 - m_2 m^{(1)}_1 < 0$.

(S3) $m_1 < 0$, $m_2 \geq 0$ *and* $m_2 m^{(2)}_1 - m_1 m^{(2)}_2 < 0$.

Remark 2.1 *The stability condition is also obtained for $m_1 = m_2 = 0$, but it is a bit more complicated (see Theorem 3.4.1 of [6]). However, this case is not excluded under the assumption (i) (see Remark 3.5).*

We shall get back to these stability conditions to see how they are related to the decay rate problem (see Corollary 4.2). Denote the unique stationary distribution of $\{\boldsymbol{L}_n\}$ by vector $\boldsymbol{\nu} = \{\nu_{nk}; n,k \in \mathbb{Z}_+\}$. Our problem is to find the asymptotic behavior of the stationary vector $\boldsymbol{\nu}$ and its marginal distributions in the direction L_1 or L_2 increased. We denote the marginal distribution in the L_1 direction by $\nu^{(1)}_n = \sum_{k=0}^{\infty} \nu_{nk}$

for $n \geq 0$. Similarly, we let $\nu_n^{(2)} = \sum_{k=0}^{\infty} \nu_{kn}$. Since coordinates are exchangeable, we mainly consider the L_1 direction in this and the next sections.

Thus, we are interested in the asymptotic behavior of ν_{nk} for each fixed k, and $\nu_n^{(1)}$ as $n \to \infty$. In general, a function $f(n)$ of $n \geq 0$ is said to have exact asymptotics $g(n)$ if there is a positive constant c such that

$$\lim_{n \to \infty} g(n)^{-1} f(n) = c,$$

and to have a rough decay rate ξ if

$$\lim_{n \to \infty} \frac{1}{n} \log f(n) = -\xi.$$

In queueing applications, it is often convenient to talk in terms of $r = e^{-\xi}$, which is referred to as a rough geometric decay rate. In general, those decay rates may not exist. So, we need to verify their existence.

As we noted in the introduction, these decay rate problems have been studied for a more general process in an arbitrary given direction by Borovkov and Mogul'skii [5] (see also [17]), but there remain several problems unsolved, which will be detailed in Section 5.1 (see, e.g., (P1)–(P4)). We will solve those problems for the double QBD process. It turns out that the rough decay rate may be different for the marginal distribution $\nu_n^{(1)}$ than ν_{nk} for each fixed k. Furthermore, the exact asymptotics $g(n)$ for ν_{nk} for each fixed k is $n^{-u} r^n$ with u either $0, \frac{1}{2}$ and $\frac{3}{2}$, while $\nu_n^{(1)}$ can decay as $n r^n$. Our approach is analytic and different from the large deviations approach in [5, 17]. However, our results can be interpreted in the context of large deviations. We make a few remarks on the relation in Section 5.2 (see Remarks 5.6 and 5.8).

Before going to detailed analysis, let us intuitively consider how the decay rates can be found. For this, let us introduce the following moment generating functions with variable $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2$, where $\mathbb{R}$ is the set of all real numbers.

$$\varphi(\boldsymbol{\theta}) = E(e^{\boldsymbol{\theta} \boldsymbol{X}}), \qquad \varphi^{(i)}(\boldsymbol{\theta}) = E(e^{\boldsymbol{\theta} \boldsymbol{X}^{(i)}}), \qquad i = 0, 1, 2.$$

Let $\tilde{\boldsymbol{L}} \equiv (\tilde{L}_1, \tilde{L}_2)$ be a random vector subject to the stationary distribution $\boldsymbol{\nu}$. Then, from the stationary equation:

$$\tilde{\boldsymbol{L}} \cong \begin{cases} \tilde{\boldsymbol{L}} + \boldsymbol{X}, & \tilde{\boldsymbol{L}} \in S_+ \setminus \partial S_+, \\ \tilde{\boldsymbol{L}} + \boldsymbol{X}^{(i)}, & \tilde{\boldsymbol{L}} \in \partial S_+^{(i)}, i = 0, 1, 2, \end{cases}$$

where $\cong$ stands for the equivalence in distribution, it follows that

$$E(e^{\boldsymbol{\theta} \tilde{\boldsymbol{L}}}) = \varphi(\boldsymbol{\theta}) E(e^{\boldsymbol{\theta} \tilde{\boldsymbol{L}}}; \tilde{\boldsymbol{L}} \in S_+ \setminus \partial S_+) + \sum_{i=0}^{2} \varphi^{(i)}(\boldsymbol{\theta}) E(e^{\boldsymbol{\theta} \tilde{\boldsymbol{L}}}; \tilde{\boldsymbol{L}} \in \partial S_+^{(i)}).$$

Rearranging terms in this equation, we have

$$(1 - \varphi(\boldsymbol{\theta})) E(e^{\boldsymbol{\theta} \tilde{\boldsymbol{L}}}) = \sum_{i=1,2} (\varphi^{(i)}(\boldsymbol{\theta}) - \varphi(\boldsymbol{\theta})) E(e^{\theta_i \tilde{L}_i} 1(\tilde{L}_i \geq 1, \tilde{L}_{3-i} = 0)) + (\varphi^{(0)}(\boldsymbol{\theta}) - \varphi(\boldsymbol{\theta})) P(\tilde{\boldsymbol{L}} = \boldsymbol{0}) \qquad (1)$$

as long as $E(e^{\boldsymbol{\theta} \tilde{\boldsymbol{L}}})$ is finite, where $1(\cdot)$ is the indicator function. From this equation, one may guess that the rough decay rate in a given direction may be obtained through the rough decay rates in the directions of coordinates. Furthermore, such a decay rate would be determined through $\varphi(\boldsymbol{\theta}) = 1$ since this $\boldsymbol{\theta}$ would be a singular point of $E(e^{\boldsymbol{\theta} \tilde{\boldsymbol{L}}})$. If this is the case, then $\varphi^{(i)}(\boldsymbol{\theta}) \leq 1$ for $i = 1$ or $i = 2$ since $\varphi^{(0)}(\boldsymbol{\theta}) > 1 = \varphi(\boldsymbol{\theta})$. In the next section, we shall see that this observation works to get a lower bound for the geometric decay rates in the directions of coordinates, i.e., the level direction in terminology of QBD processes.

We now consider the stationary distribution $\boldsymbol{\nu}$ for the decay rate in the L_1-direction. For this, it is not clear whether (1) is useful. Our first idea is to use the well known expression of the stationary distribution of the QBD process. For this, we take L_{1n} as the level and L_{2n} as the phase, and partition the state space by the level. Then, the transition probability matrix P of $\{\boldsymbol{L}_n\}$ can be written in tridiagonal block form as

$$P = \begin{pmatrix} B_0^{(1)} & B_1^{(1)} & 0 & \dots & & & & \\ A_{-1}^{(1)} & A_0^{(1)} & A_1^{(1)} & 0 & \dots & & & \\ 0 & A_{-1}^{(1)} & A_0^{(1)} & A_1^{(1)} & 0 & \dots & & \\ 0 & 0 & A_{-1}^{(1)} & A_0^{(1)} & A_1^{(1)} & 0 & \dots & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$A_k^{(1)} = \begin{pmatrix} p_{k0}^{(1)} & p_{k1}^{(1)} & 0 & \dots & & & & \\ p_{k(-1)} & p_{k0} & p_{k1} & 0 & \dots & & & \\ 0 & p_{k(-1)} & p_{k0} & p_{k1} & 0 & \dots & & \\ 0 & 0 & p_{k(-1)} & p_{k0} & p_{k1} & 0 & \dots & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \qquad k = 0, \pm 1,$$

and

$$B_k^{(1)} = \begin{pmatrix} p_{k0}^{(0)} & p_{k1}^{(0)} & 0 & \dots & & & & \\ p_{k(-1)}^{(2)} & p_{k0}^{(2)} & p_{k1}^{(2)} & 0 & \dots & & & \\ 0 & p_{k(-1)}^{(2)} & p_{k0}^{(2)} & p_{k1}^{(2)} & 0 & \dots & & \\ 0 & 0 & p_{k(-1)}^{(2)} & p_{k0}^{(2)} & p_{k1}^{(2)} & 0 & \dots & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \qquad k = 0, 1.$$

Similarly, we define $A_k^{(2)}, B_k^{(2)}$ when L_{2n} is taken as the level. Define the Markov additive process $\{\boldsymbol{L}_n^{(1)}\}$ by

$$P(\boldsymbol{L}_{n+1}^{(1)} = (m+\ell, k) | \boldsymbol{L}_n^{(1)} = (m, j)) = [A_\ell^{(1)}]_{jk}, \qquad \ell = 0, \pm 1.$$

Namely, $\{\boldsymbol{L}_n^{(1)}\}$ is obtained from $\{\boldsymbol{L}_n\}$ removing the reflection at the boundary $\{(0,k); k \geq 0\}$. Similarly, $\{\boldsymbol{L}_n^{(2)}\}$ is defined using $\{A_\ell^{(2)}; \ell = 0, \pm 1\}$. Note that $\{\boldsymbol{L}_n^{(i)}\}$ for $i = 1, 2$ can be considered as Markov additive processes with integer-valued additive components and nonnegative integer-valued background states. Throughout the paper, we use the following assumptions for simplicity.

(ii) For $i = 1, 2$, the Markov chain $\{\boldsymbol{L}_n^{(i)}\}$ is irreducible, that is, $P(X_\ell > 0)$ and $P(X_\ell < 0)$, for $\ell = 1, 2$ and $P(X_{3-i}^{(i)} > 0)$ are all positive, where X_ℓ is the ℓ-th entry of $\boldsymbol{X}$.

(iii) For $i = 1, 2$, the Markov additive kernel $\{A_\ell^{(i)}; \ell = 0, \pm 1\}$ is 1-arithmetic, where the kernel is said to be m-arithmetic if, for some $j \geq 0$, the greatest common divisor of $\{n_1 + \ldots + n_\ell; A_{n_1}^{(i)} A_{n_2}^{(i)} \ldots \times A_{n_\ell}^{(i)}(j,j) > 0, n_1, \ldots, n_\ell = 0, \pm 1\}$ is m (see, e.g., [26]).

Remark 2.2 *Condition (ii) can be removed, but special care is needed. We will discuss it in Section 7. Note that (ii) does not imply that the random walk with one step movement subject to $\boldsymbol{X}$ is irreducible. Such an example is given by $P(\boldsymbol{X} = \boldsymbol{j}) > 0$ only for $\boldsymbol{j} = (1, -1), (-1, 1), (-1, 0)$. Condition (iii) is not essential for the rough decay rate (see Remark 4.2).*

Let $\boldsymbol{\nu}_n$ be the partition of $\boldsymbol{\nu}$ for level n, that is, $\boldsymbol{\nu}_n = \{\nu_{nk}; k \in \mathbb{Z}_+\}$ for $n \geq 0$. It is well-known that there exists a nonnegative matrix R_1 uniquely determined as the minimal nonnegative solution of the matrix equation:

$$R_1 = R_1^2 A_{-1}^{(1)} + R_1 A_0^{(1)} + A_1^{(1)}, \tag{2}$$

and the stationary distribution has the following matrix geometric form.

$$\boldsymbol{\nu}_n = \boldsymbol{\nu}_1 R_1^{n-1}, \qquad n \geq 1, \tag{3}$$

From the notational uniformity, R_1 may be better written as $R^{(1)}$. However, we often use its power, for which R_1 is more convenient. We sometimes use this convention.

The matrix geometric form (3) is usually obtained when there are finitely many phases, but it is also valid for the infinitely many phases (see, e.g., [35]). Note that the transpose of R_1, denoted by $R_1^{\rm T}$, determines phase transitions at ladder epochs in a certain time reversed process of $\{\boldsymbol{L}_n^{(1)}\}$ (see Lemma 3.2 of [26]). So, $R_1^{\rm T}$ is irreducible due to (ii).

We first consider the rough geometric decay rate $r_1(k)$, equivalently, the rough decay rate $\xi_1(k)$, for ν_{nk}. Define the convergence parameter $c_p(R_1)$ of R_1 as

$$c_p(R_1) = \sup \left\{ z \geq 0; \sum_{n=0}^{\infty} z^n R_1^n < \infty \right\}.$$

Then, one may expect the reciprocal of $c_p(R_1)$ to be the rough geometric decay rate $r_1(k)$ for all k. This is certainly true when the phase space is finite. However, it is not always the case when the phase is infinite. Furthermore, we do not know even whether a rough decay rate exists at this stage. So, we introduce the lower and upper geometric decay rates $\underline{r}_1(k) \equiv e^{-\overline{\xi}_1(k)}$ and $\overline{r}_1(k) \equiv e^{-\underline{\xi}_1(k)}$ defined to be

$$\overline{\xi}_1(k) = -\limsup_{n \to \infty} \frac{1}{n} \log \nu_{nk}, \qquad \underline{\xi}_1(k) = -\liminf_{n \to \infty} \frac{1}{n} \log \nu_{nk}, \qquad k \in \mathbb{Z}_+.$$

We denote the rough decay rate of ν_{nk} for fixed k by $r_1(k)$ if it exists. Furthermore, if it is independent of k, it is denoted by r_1. As we shall see below, the convergence parameter is useful to get a lower bound for $\underline{r}_1(k)$.

Lemma 2.1 *$c_p(R_1)^{-1} \leq \underline{r}_1(k)$ for all $k \in \mathbb{Z}_+$.*

Proof. Since $R_1^{\rm T}$ is irreducible, we first pick up an arbitrary phase j, then can find some integer $m \geq 1$ for each phase k such that $[R_1^m]_{jk} > 0$. Hence, we have

$$\liminf_{n \to \infty} (\nu_{nk})^{\frac{1}{n}} \geq \lim_{n \to \infty} (\nu_{1j} [R_1^{n-m}]_{jj} [R_1^m]_{jk})^{\frac{1}{n}} = c_p(R_1)^{-1}.$$

This concludes the claimed inequality. □

In view of the matrix geometric form (3), examining the point spectrum of R_1, i.e., finding all eigenvalues, with nonnegative eigenvectors of R_1, will be useful for getting the decay rate. So, we introduce the set

$$\mathcal{V}_{R_1} = \{(z, \boldsymbol{x}); z\boldsymbol{x}R_1 = \boldsymbol{x}, z \geq 1, \boldsymbol{x} \in \mathcal{X}\},$$

where $\mathcal{X}$ is the set of all nonnegative and nonzero vectors in $\mathbb{R}^\infty$ and $\mathbb{R}$ is the set of all real numbers. We aim to identify this set through the discrete-time Markov additive process generated by $\{A_k^{(1)}; k = 0, \pm 1\}$ (see [26] for the details of this class of Markov additive processes). Note that (2) is equivalent to

$$I - A_*^{(1)}(z) = (I - zR_1)(I - (A_0^{(1)} + R_1 A_{-1}^{(1)} + z^{-1} A_{-1}^{(1)})), \qquad z \neq 0, \tag{4}$$

where $A_*^{(1)}(z)$ is defined as $A_*^{(1)}(z) = z^{-1} A_{-1}^{(1)} + A_0^{(1)} + z A_1^{(1)}$. Equation (4) is known as the Wiener-Hopf factorization (see, e.g. [26]). With the notation

$$\mathcal{V}_{A^{(1)}} = \{(z, \boldsymbol{x}); \boldsymbol{x} A_*^{(1)}(z) = \boldsymbol{x}, z \geq 1, \boldsymbol{x} > \boldsymbol{0}\},$$

we note the following facts.

Lemma 2.2 *For $z > 1$, $(z, \boldsymbol{x}) \in \mathcal{V}_{R_1}$ if and only if $(z, \boldsymbol{x}) \in \mathcal{V}_{A^{(1)}}$. Furthermore, if $\sup\{z \geq 1; (z, \boldsymbol{x}) \in \mathcal{V}_{A^{(1)}}\} = 1$, then $c_p(R_1) = 1$.*

Lemma 2.3 $c_p(R_1) = \sup\{z \geq 1; \boldsymbol{x} A_*^{(1)}(z) \leq \boldsymbol{x}, \boldsymbol{x} > \boldsymbol{0}\}$

Lemma 2.2 is obtained in [29], which is also an easy consequence of (4). We prove Lemma 2.3 in Appendix A.

Thus, we need to work on $A_*^{(1)}(z)$. To compute $A_*^{(1)}(z)$, recalling random vectors (X_1, X_2) and $(X_1^{(\ell)}, X_2^{(\ell)})$ are subject to $\{p_{ij}\}$ and $\{p_{ij}^{(\ell)}\}$ $(\ell = 1, 2)$, we introduce the following generating functions:

$$p_{i*}(v) = E(1(X_1 = i) v^{X_2}), \qquad p_{*j}(u) = E(u^{X_1} 1(X_2 = j)),$$
$$p_{i*}^{(\ell)}(v) = E(1(X_1^{(\ell)} = i) v^{X_2^{(\ell)}}), \qquad p_{*j}^{(\ell)}(u) = E(u^{X_1^{(\ell)}} 1(X_2^{(\ell)} = j)), \qquad \ell = 0, 1, 2.$$

Then, $A_*^{(1)}(z)$ has the following QBD structure.

$$A_*^{(1)}(z) = \begin{pmatrix} p_{*0}^{(1)}(z) & p_{*1}^{(1)}(z) & 0 & \dots & & & & \\ p_{*(-1)}(z) & p_{*0}(z) & p_{*1}(z) & 0 & \dots & & & \\ 0 & p_{*(-1)}(z) & p_{*0}(z) & p_{*1}(z) & 0 & \dots & & \\ 0 & 0 & p_{*(-1)}(z) & p_{*0}(z) & p_{*1}(z) & 0 & \dots & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{5}$$

Since L_{1n} and L_{2n} will have symmetric roles in our arguments, we can exchange their roles. That is, we take L_{2n} and L_{1n} as the level and the phase, respectively. Since it will be not hard to convert results on the original model to this case, we only consider the case that L_{1n} is the level in the next section.

3. Basic tools for finding the geometric decay. Before working on $A_*^{(1)}(z)$, we sort available sufficient conditions for asymptotically exact geometric decay. To this end, we refer to the results in [26], and extend those obtained by Li, Miyazawa and Zhao [16]. To present them, it is convenient to use the following terminology due to Seneta [32]. A nonnegative square matrix T is said to be z-positive for $z > 0$ if zT has nonnegative nonzero left and right invariant vectors $\boldsymbol{s}$ and $\boldsymbol{t}$, respectively, such that $\boldsymbol{st} < \infty$ (see Theorem 6.4 of [32]). Throughout this section, we assume conditions (i), (ii) and (iii).

It is shown in [16] that $A_*^{(1)}(z)$ is 1-positive if and only if R_1 is z-positive. Then, the following result is obtained in [16] (see also [26]).

Proposition 3.1 (Theorem 2.1 of [16]) *Assume the following two conditions.*

(F1) There exist an $z > 1$, a positive row vector $\boldsymbol{x}$ and a positive column vector $\boldsymbol{y}$ such that

$$\boldsymbol{x}A_*^{(1)}(z) = \boldsymbol{x}, \tag{6}$$

$$A_*^{(1)}(z)\boldsymbol{y} = \boldsymbol{y}, \tag{7}$$

(F2) $\boldsymbol{xy} < \infty$ for $\boldsymbol{x}$ and $\boldsymbol{y}$ in (F1),

then we have

$$\lim_{n\to\infty} z^n \boldsymbol{\nu}_n = \boldsymbol{c}, \tag{8}$$

for $\boldsymbol{c} = \dfrac{z\boldsymbol{\nu}_1\boldsymbol{q}}{\boldsymbol{xq}}\boldsymbol{x} > \boldsymbol{0}$, where $\boldsymbol{q}$ is the right invariant vector of zR_1 and limit is taken in component-wise. The $\boldsymbol{c}$ may be infinite, but is finite if the condition

(F3) $\boldsymbol{\nu}_1\boldsymbol{q} < \infty$ for the right eigenvector $\boldsymbol{q}$ of zR_1

is satisfied. The condition $\boldsymbol{\nu}_1\boldsymbol{y} < \infty$ is sufficient for (F3). Thus, (F1)–(F3) imply that z^{-1} is the asymptotically exact geometric decay rate of ν_{nk} for each fixed k.

The next proposition, which is an extension of Theorem 2.2.1 of [16], will be a key for identifying the decay rate. We prove it in Appendix B.

Proposition 3.2 *Let $\boldsymbol{x} = (x_k)$ be a positive vector satisfying (6) for some $z > 1$. Define $\underline{d}(z, \boldsymbol{x})$ and $\overline{d}(z, \boldsymbol{x}) \geq 0$ as*

$$\underline{d}(z, \boldsymbol{x}) = \liminf_{k\to\infty} \frac{\nu_{1k}}{x_k}, \qquad \overline{d}(z, \boldsymbol{x}) = \limsup_{k\to\infty} \frac{\nu_{1k}}{x_k}.$$

Assume that $\overline{d}(z, \boldsymbol{x})$ is finite. Then, for any nonnegative column vector $\boldsymbol{u}$ satisfying $\boldsymbol{xu} < \infty$, we have

(3a) If (F2) holds, then, for some $d^\dagger > 0$,

$$\lim_{n\to\infty} z^n \boldsymbol{\nu}_n \boldsymbol{u} = zd^\dagger \boldsymbol{xu}. \tag{9}$$

(3b) If (F2) does not hold, then there are nonegative and finite $\underline{d}^\dagger(\boldsymbol{x})$ and $\overline{d}^\dagger(\boldsymbol{x})$ such that

$$z\underline{d}^\dagger(\boldsymbol{x})\boldsymbol{x}\boldsymbol{u} \le \liminf_{n\to\infty} z^n \boldsymbol{\nu}_n \boldsymbol{u} \le \limsup_{n\to\infty} z^n \boldsymbol{\nu}_n \boldsymbol{u} \le z\overline{d}^\dagger(\boldsymbol{x})\boldsymbol{x}\boldsymbol{u}. \tag{10}$$

Hence, the rough upper geometric decay rate $\overline{r}_1(k)$ is bounded by z^{-1} for all $k \in \mathbb{Z}_+$. In particular, if $\underline{d}(z,\boldsymbol{x}) > 0$, then the rough geometric decay rate r_1 exists and given by $r_1 = z^{-1}$. Furthermore, if $\underline{d}^\dagger(\boldsymbol{x}) = \overline{d}^\dagger(\boldsymbol{x})$, then we have (9) for $d^\dagger \equiv \underline{d}^\dagger(\boldsymbol{x}) > 0$.

REMARK 3.1 *If $\underline{d}(z,\boldsymbol{x}) > 0$, then $\boldsymbol{x}$ is summable, i.e., $\boldsymbol{x}\mathbf{1} < \infty$. Hence, we can take $\boldsymbol{u} = \mathbf{1}$, and (3a) or (3b) with $\underline{d}^\dagger(\boldsymbol{x}) = \overline{d}^\dagger(\boldsymbol{x})$ implies that the asymptotic decay of the marginal stationary distribution for L_1 is exactly geometric with rate z^{-1}.*

REMARK 3.2 *(3b) obviously includes the case that $\boldsymbol{x} = \boldsymbol{\nu}_1$. By (3), this implies that $\boldsymbol{\nu}_n = z^{-(n-1)}\boldsymbol{x}$. So, the stronger version of (10) is obtained. However, this occurs in a very special way, and it is generally not the case for the double QBD process (see [13]).*

It is notable that the decay rates obtained in Propositions 3.1 and 3.2 are independent of the phase. This may not be surprising since the state transitions are homogeneous inside the state space S_+. However, both require some information on the marginal probability vector $\boldsymbol{\nu}_1$. So, it is crucial to get this information to use those results. We shall return to this point in Section 4.

We now consider how to identify $\mathcal{V}_{A^{(1)}}$ using the specific form (5) of $A_*^{(1)}(z)$. Noting the assumption (ii), let $w_1(z)$ and $w_2(z)$ be the solutions of the quadratic equation of $w(z)$

$$p_{*(-1)}(z)w^2(z) + p_{*0}(z)w(z) + p_{*1}(z) = w(z) \tag{11}$$

for each fixed z, then the i-th entries x_i and y_i of $\boldsymbol{x}$ and $\boldsymbol{y}$ satisfying (6) and (7) with this z, respectively, must have the form

$$x_n = x_1 w_1^{n-1}(z) + (x_2 - x_1 w_1(z)) \sum_{\ell=0}^{n-2} w_1^\ell(z) w_2^{n-2-\ell}(z) \qquad n \ge 1, \tag{12}$$

and

$$y_n = y_0 w_2^{-n}(z) + (y_1 - y_0 w_2^{-1}(z)) \sum_{\ell=0}^{n-1} w_1^{-n+1+\ell}(z) w_2^{-\ell}(z) \qquad n \ge 0, \tag{13}$$

where empty sums are assumed to vanish. These two expressions are elementary, but will be a key to finding positive invariant vectors. It is not hard to see that $\boldsymbol{x}$ and $\boldsymbol{y}$ can not be positive if the $w_i(z)$'s are complex numbers (see page 529 of [27] for the details). It then follows from the fact that $w_1(z)$ and $w_2(z)$ are roots of equation (11) that they must both be positive. Thus, we can see that $x_2 - x_1 w_1(z) \ge 0$ is necessary and sufficient for x_n to be positive for $n \ge 2$.

Since the first and second entries of the vector equations in (6) are

$$p_{*0}^{(1)}(z)x_0 + p_{*(-1)}(z)x_1 = x_0,$$
$$p_{*1}^{(1)}(z)x_0 + p_{*0}(z)x_1 + p_{*(-1)}(z)x_2 = x_1,$$

it can be seen after some manipulation that $x_2 - x_1 w_1(z) \ge 0$ is equivalent to

$$p_{*0}^{(1)}(z)p_{*1}(z) + p_{*(-1)}(z)p_{*1}^{(1)}(z)w_1(z) \le p_{*1}(z), \tag{14}$$

Hence, $\boldsymbol{x}$ of (12) is positive if and only if (14) holds. Similarly, from (7), we have

$$p_{*0}^{(1)}(z)y_0 + p_{*1}^{(1)}(z)y_1 = y_0.$$

Hence, $\boldsymbol{y}$ of (13) is positive if and only if

$$p_{*0}^{(1)}(z) + p_{*1}^{(1)}(z)w_2^{-1}(z) \le 1. \tag{15}$$

Note that (14) is equivalent to (15) since $w_1(z)w_2(z) = \frac{p_{*1}(z)}{p_{*(-1)}(z)}$ from (11). We also note that the role of $w_1(z)$ and $w_2(z)$ can be exchanged.

We can apply a similar argument for a subinvariant positive vector $\boldsymbol{x}$ of $A^{(1)}_*(z)$, i.e., $\boldsymbol{x}A^{(1)}_*(z) \le \boldsymbol{x}$. In this argument, such a positive $\boldsymbol{x}$ exists if and only if (14), equivalently, (15) is satisfied, and (12) is replaced by

$$x_n \le x_1 w_1^{n-1}(z) + (x_2 - x_1 w_1(z)) \sum_{\ell=0}^{n-2} w_1^{\ell}(z) w_2^{n-2-\ell}(z) \qquad n \ge 1.$$

Hence, whenever there exists a positive subinvariant vector $\boldsymbol{x}$, there also exists a positive invariant vector $\boldsymbol{x}$. This with Lemmas 2.2 and 2.3 leads to the conclusion

Lemma 3.1 $c_p(R_1) = \sup\{z \ge 1; \boldsymbol{x}A^{(1)}_*(z) = \boldsymbol{x}, \boldsymbol{x} > \boldsymbol{0}\}$.

Recalling the notation $\varphi(\boldsymbol{\theta}) = E(e^{\theta_1 X_1 + \theta_2 X_2})$ and $\varphi^{(i)}(\boldsymbol{\theta}) = E(e^{\theta_1 X_1^{(i)} + \theta_2 X_2^{(i)}})$ for $i = 1, 2$, the above observations lead to the following results.

Theorem 3.1 *Assume (i) and (ii), and let $\mathcal{D}_1$ denote the subset of all $\boldsymbol{\theta} = (\theta_1, \theta_2)$ in $\mathbb{R}^2_1$ such that*

$$\varphi(\boldsymbol{\theta}) = 1, \tag{16}$$
$$\varphi^{(1)}(\boldsymbol{\theta}) \le 1, \tag{17}$$
$$\theta_1 \ge 0, \theta_2 \in \mathbb{R}.$$

Then, there exists a one to one relationship between a $(\theta_1, \theta_2) \in \mathcal{D}_1$ and $(z, \boldsymbol{x}) \in \mathcal{V}_{A^{(1)}}$, equivalently $(z, \boldsymbol{x}) \in \mathcal{V}_{R_1}$. Furthermore, we have the following facts.

(3c) For each θ_1 for which there exists θ_2 with $(\theta_1, \theta_2) \in \mathcal{D}_1$, $(z, \boldsymbol{x}) \in \mathcal{V}_{A^{(1)}}$ is given by $z = e^{\theta_1}$ and $\boldsymbol{x} = \{x_n\}$ with

$$x_n = \begin{cases} c_1 e^{-\underline{\theta}_2(n-1)} + c_2 e^{-\overline{\theta}_2(n-1)}, & \underline{\theta}_2 \ne \overline{\theta}_2, \\ (c'_1 + c'_2(n-1))\, e^{-\underline{\theta}_2(n-2)}, & \underline{\theta}_2 = \overline{\theta}_2, \end{cases} \qquad n \ge 1, \tag{18}$$

where $\underline{\theta}_2, \overline{\theta}_2$ are the two solutions of (16) for the given θ_1 such that $\underline{\theta}_2 \le \overline{\theta}_2$, and c_i, c'_i are given by

$$c_1 = \frac{x_2 - x_1 e^{-\overline{\theta}_2}}{e^{-\underline{\theta}_2} - e^{-\overline{\theta}_2}}, \quad c_2 = \frac{x_1 e^{-\underline{\theta}_2} - x_2}{e^{-\underline{\theta}_2} - e^{-\overline{\theta}_2}}, \quad c'_1 = x_1 e^{-\underline{\theta}_2}, \quad c'_2 = x_2 - x_1 e^{-\underline{\theta}_2},$$

where x_1 and x_2 are determined by the first two entries of (6). Furthermore, both of c_1 and c_2 are not zero only if strict inequality holds in (17).

(3d) The convergence parameter $c_p(R_1)$ is obtained as the supremum of e^{θ_1} over $\mathcal{D}_1$.

(3e) R_1 is z-positive if and only if equality in (17) holds with $\theta_1 = \log c_p(R_1)$ and $E(X_2 e^{\theta_1 X_1 + \theta_2 X_2}) \ne 0$, in which case $z = c_p(R_1)$.

(3f) The vector $\boldsymbol{x}$ in (3c) is summable, i.e., the sum of its entries is finite, if and only if either the solution $\overline{\theta}_2$ of (16) is positive for the case that $x_n = c_2 e^{-\overline{\theta}_2(n-1)}$, or the solution $\underline{\theta}_2 > 0$ for the other case.

Remark 3.3 *The irreducibility condition (ii) guarantees the existence of $\underline{\theta}_2$ and $\overline{\theta}_2$ for each $\boldsymbol{\theta}$ in (3c). However, the set of $\boldsymbol{\theta}$ satisfying (16) may not be a closed loop although it is the boundary of a convex set (see Figure 2 below).*

Remark 3.4 *Since R_1 and $A^{(1)}_*(e^{\theta_1})$ have the same left eigenvector due to (4), (3d) of this theorem can be obtained from Proposition 3 of [9] and Proposition 10 of [18].*

Remark 3.5 *If $m_1 = m_2 = 0$, then (16) may be reduced to a straight line. However, this case is still included in Theorem 3.1.*

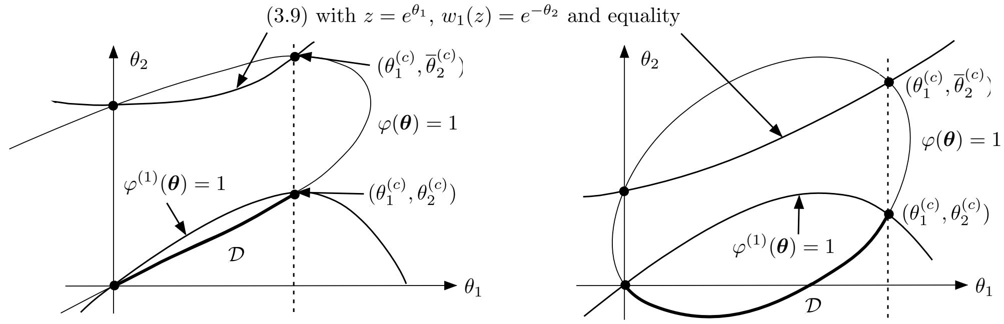

Figure 2: Typical shapes of set $\mathcal{D}_1$, where $e^{\theta_1^{(c)}}$ is the convergence parameter of R_1.

Proof of Theorem 3.1. By Lemma 2.3, the first part of this theorem is equivalent to showing that the matrix generating function $A_*^{(1)}(z)$ of the Markov additive kernel has a positive left invariant vector $\boldsymbol{x}$ for $z > 1$ that has a one to one mapping to a $(\theta_1, \theta_2) \in \mathcal{D}_1$. The conditions (16) and (17) for this are equivalent to (11) with $w(z) = w_2(z)$ and (15), respectively, if we let $\theta_1 = \log z$ and $\theta_2 = -\log w_2(z)$. From the arguments before this theorem, these conditions are equivalent to the existence of the positive left invariant vector $\boldsymbol{y}$ of $A_*^{(1)}(z)$, where $\boldsymbol{y}$ is given by (13). Since the positive right invariant vector of $A_*^{(1)}(z)$ exists if and only if the corresponding positive left invariant vector exists, the conditions are equivalent to that $A_*^{(1)}(z)$ has the left positive invariant vector $\boldsymbol{x}$. Thus, we have proved the first part.

We next prove (3c)–(3f). Statement (3c) is easily obtained from $\boldsymbol{x}$ of (12), where we have used the fact that strict inequality in (14) is equivalent to strict inequality in (15). Statement (3d) is obtained by Lemma 2.3. The remaining statements (3e) and (3f) obviously follow from (3c). □

We now consider the decay rate of the $\boldsymbol{\nu}_n$. For this, there are the following three scenarios. Let $\boldsymbol{y}$ be the positive right invariant vector of $A_*^{(1)}(z)$.

(R1) R_1 is z-positive and $\boldsymbol{\nu}_1 \boldsymbol{y} < \infty$.
(R2) R_1 is z-positive and $\boldsymbol{\nu}_1 \boldsymbol{y} = \infty$.
(R3) R_1 is not z-positive.

For scenario (R1), we have that the asymptotic decay is exactly geometric by Proposition 3.1. However, condition $\boldsymbol{\nu}_1 \boldsymbol{y} < \infty$ is not easy to verify even in this case. For all the scenarios, we shall use the following fact.

Corollary 3.1 *Assume (i), (ii) and (iii), and define β as*

$$\beta = \sup \left\{ \theta_1; \exists \theta_2 \text{ such that } \limsup_{n \to \infty} \nu_{1n} e^{\theta_2 n} < \infty \text{ and } (\theta_1, \theta_2) \in \mathcal{D}_1 \right\},$$

then the rough upper decay rate $\overline{r}_1(k)$ is bounded by $e^{-\beta}$ for all $k \in \mathbb{Z}_+$. In particular, if $\beta = \log c_p(R_1)$, then the rough decay rate r_1 is given by $r_1 = e^{-\beta}$, independently of the phase. Furthermore, if the asymptotic decay of ν_{1n} is exactly geometric as $n \to \infty$, then the stationary level distribution $\boldsymbol{\nu}_n$ also asymptotically decays in the exact geometric form with rate $e^{-\beta}$.

Since the proof of this corollary is technical, we defer it to Appendix C. Corollary 3.1 may look not so sharp since it only gives an upper bound for the rough decay rates. Furthermore, it requires some asymptotic information on ν_{1n}. Nevertheless, combining it with the lower bound of Lemma 2.1 will determine the decay rates. This will be discussed in the next section. Thus, Corollary 3.1 acts as a bridge between the tail asymptotic behavior on the two boundary faces.

4. Characterization of the decay rates. As we have seen in Propositions 3.1 and 3.2, the asymptotic behavior in one direction requires some information on the behavior in the other direction. In this section, we simultaneously consider both directions to solve the decay rate problem. For this, we need to exchange the level and the phase, that is, to take L_{2n} and L_{1n} as level and phase, respectively. To avoid possible confusion, we keep the order of L_{1n} and L_{2n} for the state description. Thus, when the level direction is exchanged, the asymptotics of $\boldsymbol{\nu}^{\dagger}_n = \{\nu_{\ell n}; \ell \in \mathbb{Z}_+\}$ as $n \to \infty$ is of our interest. We refer to this asymptotics as in the L_2-direction while the original asymptotics as in the L_1-direction. In the previous sections, we have been only concerned with the L_1-direction. Obviously, the results there can be converted for the L_2-direction.

To present those results, we define $\mathcal{D}_2$ similarly to $\mathcal{D}_1$, and use η_i instead of θ_i when L_{2n} is taken as the level. That is, $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined as

$$\mathcal{D}_1 = \{\boldsymbol{\theta} \in \mathbb{R}^2; \varphi(\boldsymbol{\theta}) = 1, \varphi^{(1)}(\boldsymbol{\theta}) \le 1, \theta_1 \ge 0, \theta_2 \in \mathbb{R}\}$$
$$\mathcal{D}_2 = \{\boldsymbol{\eta} \in \mathbb{R}^2; \varphi(\boldsymbol{\eta}) = 1, \varphi^{(2)}(\boldsymbol{\eta}) \le 1, \eta_2 \ge 0, \eta_1 \in \mathbb{R}\},$$

The rough decay rates in the coordinate directions are basically known under some extra conditions (see Theorem 3.6.3 of [17]). The following characterization not only removes the extra conditions but also provides a new graphical interpretation for them.

Theorem 4.1 *For the double QBD process satisfying the assumptions (i)–(iii), define α_i for $i = 1, 2$ as*

$$\alpha_1 = \sup\{\theta_1; \eta_1 \le \theta_1, \theta_2 \le \eta_2, (\theta_1, \theta_2) \in \mathcal{D}_1, (\eta_1, \eta_2) \in \mathcal{D}_2\}, \tag{19}$$
$$\alpha_2 = \sup\{\eta_2; \eta_1 \le \theta_1, \theta_2 \le \eta_2, (\theta_1, \theta_2) \in \mathcal{D}_1, (\eta_1, \eta_2) \in \mathcal{D}_2\}. \tag{20}$$

Then, α_1 and α_2 are the rough decay rates, that is, $e^{-\alpha_1}$ and $e^{-\alpha_2}$ are the rough geometric decay rates, of ν_{nk} and ν_{kn}, respectively, as $n \to \infty$ for each fixed k.

Remark 4.1 *The segment connecting the two points on $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, corresponding to α_1 and α_2 must have non positive slope. This is a convenient way of identifying those points.*

Remark 4.2 *For simplicity, we have assumed the non-arithmetic condition (iii), but it can be removed. For this, we only need to update Proposition 3.1 and (3a) of Proposition 3.2 for the arithmetic case. Note that $\{A^{(1)}_\ell; \ell = 0, \pm 1\}$ is d-arithmetic if and only if R is d-periodic (see Remark 4.4 of [26]). Hence, if $\{A^{(1)}_\ell; \ell = 0, \pm 1\}$ is d-arithmetic, then we can prove (8) for dn instead of n, so the rough decay rate is unchanged.*

Before proving this theorem, we identify the locations of the α_i on $\mathcal{D}_i$ for $i = 1, 2$, which will give us explicit form of the rough decay rates. For this, we For convenience, we put $\mathcal{D}_0 = \{\boldsymbol{\theta} \in \mathbb{R}^2; \varphi(\boldsymbol{\theta}) = 1\}$, and introduce the following points concerning the convergence parameters of R_1 and R_2.

$$\boldsymbol{\theta}^{(c)} \equiv (\theta^{(c)}_1, \theta^{(c)}_2) = \arg\max_{\boldsymbol{\theta} \in \mathcal{D}_1} \theta_1, \qquad \overline{\theta}^{(c)}_2 = \max\{\theta_2; (\theta^{(c)}_1, \theta_2) \in \mathcal{D}_0\},$$
$$\boldsymbol{\eta}^{(c)} \equiv (\eta^{(c)}_1, \eta^{(c)}_2) = \arg\max_{\boldsymbol{\eta} \in \mathcal{D}_2} \eta_2, \qquad \overline{\eta}^{(c)}_1 = \max\{\eta_1; (\eta_1, \eta^{(c)}_2) \in \mathcal{D}_0\},$$

where the maximum and minimum exist since $\mathcal{D}_1$ and $\mathcal{D}_2$ are bounded closed sets. We classify the configuration of points $\boldsymbol{\theta}^{(c)}$ and $\boldsymbol{\eta}^{(c)}$ according to the following conditions.

$$\text{(C1)}\ \ \eta^{(c)}_1 < \theta^{(c)}_1 \text{ and } \theta^{(c)}_2 < \eta^{(c)}_2, \qquad \text{(C2)}\ \ \eta^{(c)}_1 < \theta^{(c)}_1 \text{ and } \eta^{(c)}_2 \le \theta^{(c)}_2$$
$$\text{(C3)}\ \ \theta^{(c)}_1 \le \eta^{(c)}_1 \text{ and } \theta^{(c)}_2 < \eta^{(c)}_2, \qquad \text{(C4)}\ \ \theta^{(c)}_1 \le \eta^{(c)}_1 \text{ and } \eta^{(c)}_2 \le \theta^{(c)}_2.$$

The case (C4) is impossible since $\theta^{(c)}_1 \le \eta^{(c)}_1$ implies that $\eta^{(c)}_2 > \theta^{(c)}_2$ due to the convexity of the set with boundary (16). This classification is illustrated in Figures 3 and 4, where $\theta^{\max}_i, \eta^{\max}_i$ will be defined in Theorem 4.2. We will see that this new classification is quite useful for finding not only the rough decay rate but also the exact asymptotics.

Corollary 4.1 *The rough decay rates α_1 and α_2 in Theorem 4.1 are obtained as*

$$(\alpha_1, \alpha_2) = \begin{cases} (\theta^{(c)}_1, \eta^{(c)}_2), & \text{if (C1) holds,} \\ (\overline{\eta}^{(c)}_1, \eta^{(c)}_2), & \text{if (C2) holds,} \\ (\theta^{(c)}_1, \overline{\theta}^{(c)}_2), & \text{if (C3) holds.} \end{cases} \tag{21}$$

Remark 4.3 *At least one of e^{α_i} is equal to the convergence parameter of R_i. In Section 5.2, we will see that classifications (C1), (C2) and (C3) correspond with how to choose an optimal path in the large deviations approach (see Remark 5.8).*

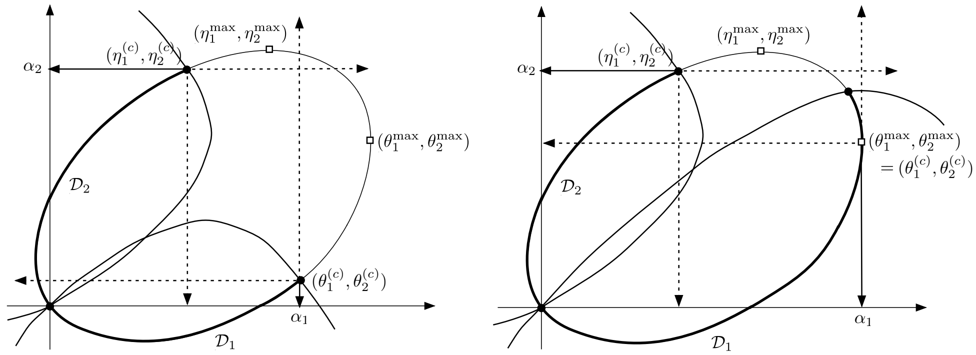

Figure 3: Case (C1) for $\theta_1^{(c)} < \theta_1^{\max}$ and for $\theta_1^{(c)} = \theta_1^{\max}$

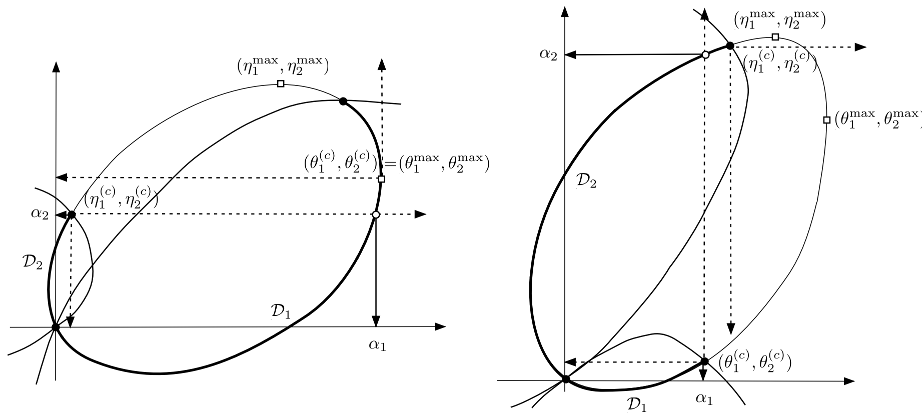

Figure 4: Cases (C2) and (C3)

Proof of Corollary 4.1. Note that $\alpha_1 \le \theta_1^{(c)}$ and $\alpha_2 \le \eta_2^{(c)}$. If (C1) holds, then $\boldsymbol{\theta} = (\theta_1^{(c)}, \theta_2^{(c)})$ and $\boldsymbol{\eta} = (\eta_1^{(c)}, \eta_2^{(c)})$ satisfy the conditions in (19) and (20), respectively. Hence, we have $\alpha_1 = \theta_1^{(c)}$ and $\alpha_2 = \eta_2^{(c)}$ by the maximal property of $\theta_1^{(c)}$ and $\eta_2^{(c)}$. Next suppose that (C2) holds, then $(\overline{\eta}_1^{(c)}, \eta_2^{(c)}) \in \mathcal{D}_1$, so we put $\boldsymbol{\theta} = (\overline{\eta}_1^{(c)}, \eta_2^{(c)})$ and $\boldsymbol{\eta} = (\eta_1^{(c)}, \eta_2^{(c)})$. Then, we can see that $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$ satisfy the conditions in (19) and (20). Hence, $\alpha_2 = \eta_2^{(c)}$. Since $\theta_2 \le \eta_2 \le \eta_2^{(c)}$ in (19), θ_1 is bounded by $\overline{\eta}_1^{(c)}$. Hence, $\alpha_1 = \overline{\eta}_1^{(c)}$. Finally, the results for (C3) are obtained by exchanging the roles of (θ_1, θ_2) and (η_1, η_2). □

Proof of Theorem 4.1. Let $\overline{\xi}_1$ and $\overline{\xi}_2$ be the upper rough decay rates of ν_{n1} and ν_{1n}, respectively. That is, $\overline{r}_1(1) = e^{-\overline{\xi}_1}$ and $\overline{r}_2(1) = e^{-\overline{\xi}_2}$ are the rough upper geometric decay rates. If $0 \le \beta_1 \le \overline{\xi}_1$ and $0 \le \beta_2 \le \overline{\xi}_2$, then $\overline{r}_1(1) \le e^{-\beta_1}$ and $\overline{r}_2(1) \le e^{-\beta_2}$, so Corollary 3.1 leads

$$f_1(\beta_2) \equiv \max\{\theta_1; \theta_2 \le \beta_2, (\theta_1, \theta_2) \in \mathcal{D}_1\} \le \overline{\xi}_1,$$
$$f_2(\beta_1) \equiv \max\{\theta_2; \theta_1 \le \beta_1, (\theta_1, \theta_2) \in \mathcal{D}_2\} \le \overline{\xi}_2.$$

We next inductively define the following sequences $\beta_1^{(n)}$ and $\beta_2^{(n)}$ for $n = 0, 1, \ldots$ with $\beta_1^{(0)} = \beta_2^{(0)} = 0$.

$$\beta_1^{(n)} = f_1(\beta_2^{(n-1)}), \qquad \beta_2^{(n)} = f_2(\beta_1^{(n)}), \qquad n = 1, 2, \ldots.$$

Obviously, f_i is a nondecreasing functions for $i = 1, 2$, $\beta_1^{(n)} = f_1(f_2(\beta_1^{(n-1)}))$ and $\beta_2^{(n)} = f_2(f_1(\beta_2^{(n-1)}))$. Since $\beta_i^{(n)}$ is nondecreasing in n for each $i = 1, 2$, we can inductively see that

$$\beta_1^{(n)} \le \overline{\xi}_1, \qquad \beta_2^{(n)} \le \overline{\xi}_2, \qquad n = 0, 1, \ldots. \tag{22}$$

From the definitions, we have that $\theta_2 \le \beta_2^{(n-1)} \le \beta_2^{(n)}$ for $(\beta_1^{(n)}, \theta_2) \in \mathcal{D}_1$ and $\eta_1 \le \beta_1^{(n)}$ for $(\eta_1, \beta_2^{(n)})$ since $\beta_2^{(n)}$ is nondecreasing in n. Hence, $\beta_i^{(n)} \le \log \alpha_i$ for $i = 1, 2$. Again, from the fact that $\beta_i^{(n)}$ is nondecreasing in n for each $i = 1, 2$, we have

$$\beta_i^{(\infty)} \equiv \lim_{n \to \infty} \beta_i^{(n)} \le \alpha_i, \qquad i = 1, 2. \tag{23}$$

Suppose either one of $\beta_i^{(\infty)}$ is strictly less than α_i. Say, $\beta_1^{(\infty)} < \alpha_1$. Let $\underline{\theta}_2^{(\infty)} = \min\{\theta_2; (\beta_1^{(\infty)}, \theta_2) \in \mathcal{D}_1\}$, $\overline{\theta}_2^{(\infty)} = \max\{\theta_2; (\beta_1^{(\infty)}, \theta_2) \in \mathcal{D}_2\}$. Then, $\beta_2^{(\infty)} = \min(\overline{\theta}_2^{(\infty)}, \alpha_2)$. If $\overline{\theta}_2^{(\infty)} < \alpha_2$, then we can find $(\theta_1, \theta_2) \in \mathcal{D}_1$ such that $\underline{\theta}_2^{(\infty)} < \theta_2 \le \overline{\theta}_2^{(\infty)}$. This means that $\beta_1^{(\infty)} < f_1(\beta_2^{(\infty)})$, which contradicts that $\beta_1^{(\infty)} = f_1(f_2(\beta_1^{(\infty)}))$. On the other hand, if $\overline{\theta}_2^{(\infty)} \ge \alpha_2$, then $\beta_2^{(\infty)} = \alpha_2$, and $\underline{\theta}_2^{(\infty)} \le \alpha_2$. If $\underline{\theta}_2^{(\infty)} < \alpha_2$, then $\beta_1^{(\infty)} < f_1(\beta_2^{(\infty)})$, which is impossible. Hence, $\underline{\theta}_2^{(\infty)} = \alpha_2$, so we must have that $\beta_1^{(\infty)} = \alpha_1$, which contradicts the assumption that $\beta_1^{(\infty)} < \alpha_1$. Consequently, we must have equalities in both inequalities (23). Hence, (22) implies

$$\alpha_1 \le \overline{\xi}_1, \qquad \alpha_2 \le \overline{\xi}_2.$$

Combining this with Lemma 2.1, we have

$$c_p(R_1)^{-1} \le \underline{r}_1(1) \le \overline{r}_1(1) \le e^{-\alpha_1}, \tag{24}$$
$$c_p(R_2)^{-1} \le \underline{r}_2(1) \le \overline{r}_2(1) \le e^{-\alpha_2}. \tag{25}$$

On the other hand, as we can see from Figures 3 and 4, either one of $e^{\alpha_1} = c_p(R_1)$ or $e^{\alpha_2} = c_p(R_2)$ holds. Hence, we have the rough decay rate at least for one direction. Say this one is L_1, then we have $r_1(1) = e^{-\alpha_1}$. If either (C1) with $\overline{\eta}_1^{(c)} \le \alpha_1$, (C2) or (C3) holds, then this and Proposition 3.2 yield that $\boldsymbol{\nu}_n^\dagger$ has the rough decay rate $e^{-\alpha_2}$. Otherwise, if (C1) with $\alpha_1 < \overline{\eta}_1^{(c)}$ holds, then Proposition 3.1 yields the same result. Both imply in turn that the rough decay rate of $\boldsymbol{\nu}_n$ is $e^{-\alpha_1}$ □

One may wonder whether the α_1 and α_2 of (19) and (20) (equivalently, of (21)) provide a condition for the existence of the stationary distribution. For example, does the condition that $\alpha_1 > 1$ and $\alpha_2 > 1$ imply stability ? Unfortunately, this is not true, but we can say that it is almost true.

Corollary 4.2 *For the double QBD process that is irreducible and satisfies (ii) and (iii), if $m_1 < 0$ or $m_2 < 0$, then it is stable if and only if $\alpha_1 > 0$ and $\alpha_2 > 0$. Otherwise, if $m_1 \ge 0$ and $m_2 \ge 0$, then this is not always the case.*

We prove this corollary in Appendix D. It allows us to compute α_i without assuming the stability once we know at least one of m_i's is negative. In other words, we can use (19) and (20) of Corollary 4.1 to characterize the stability.

As for the marginal distributions, we only consider $\nu_n^{(1)} \equiv \boldsymbol{\nu}_n \mathbf{1}$ since the results are symmetric for $\nu_n^{(2)} \equiv \boldsymbol{\nu}_n^\dagger \mathbf{1}$.

Corollary 4.3 *Under the assumptions of Theorem 4.1, assume that $\alpha_2 > 0$, and let $\tilde{\alpha}_1$ be a maximum solution of $\varphi((\tilde{\alpha}_1, 0)) = 1$. Then, we have the following cases.*

- *(4a) If $\overline{\theta}_2^{(c)} \ge 0$, then the rough decay rate of $\nu_n^{(1)}$ is α_1.*
- *(4b) If $\overline{\theta}_2^{(c)} < 0$, then $\tilde{\alpha}_1 < \alpha_1$ and we have two cases.*
  - *(4b1) If $\varphi^{(1)}((\tilde{\alpha}_1, 0)) \ge 1$, then the rough decay rate of $\nu_n^{(1)}$ is $\tilde{\alpha}_1$.*
  - *(4b2) If $\varphi^{(1)}((\tilde{\alpha}_1, 0)) < 1$, then the rough decay rate of $\nu_n^{(1)}$ is either α_1 or $\tilde{\alpha}_1$.*

Remark 4.4 *It is notable that taking a marginal distribution may change the decay rate. There are some reasons for this. For example, it happens that $\overline{\theta}_2^{(c)} < 0$ when $m_1 \le 0$ and $m_2 > 0$ (see Figure 6 in Appendix D). That is, it is most likely that L_{2n} takes a large value inside the positive quadrant, which would change the decay rate. We next consider where (C1) holds and $\overline{\theta}_2^{(c)} < 0$. Note that $\boldsymbol{x}\mathbf{1} = \infty$ for the invariant vector $\boldsymbol{x}$ for $z = e^{\theta_1^{(c)}}$ in Theorem 3.1. Since $\theta_1^{(c)} = \alpha_1$ in this case, from (8) of Proposition 3.1 and Fatou's lemma, we have*

$$\liminf_{n\to\infty} e^{\alpha_1(n-1)} \nu_n^{(1)} \ge \frac{\boldsymbol{\nu}_1 \boldsymbol{q}}{\boldsymbol{x}\boldsymbol{q}} \boldsymbol{x}\mathbf{1} = \infty. \tag{26}$$

Remark 4.5 $\varphi^{(1)}((\tilde{\alpha}_1, 0)) \ge 1$ *is satisfied for (4b) if $\theta_1^{(c)} \ne \max\{\theta_1; (\theta_1, \theta_2) \in \mathcal{D}_0\}$.*

Proof. If $\alpha_1 = 0$, then this rough decay rate must also be 0. So, we only consider the case that $\alpha_1 > 0$. For case (C2), we can apply (3b) of Proposition 3.2 with $\boldsymbol{u} = \mathbf{1}$ since $\inf\{\theta_2; (\alpha_1, \theta_2) \in \mathcal{D}_1\} = \alpha_2 > 0$, so $\overline{\theta}_2^{(c)} > 0$ and $\nu_n^{(1)}$ has the asymptotically exact decay rate α_1. We now suppose that (C1) or (C3) holds. In this case, we always have $\alpha_1 = \theta_1^{(c)}$. Denote the moment generating function of $\{\nu_n^{(1)}\}$ by $\tilde{\nu}^{(1)}(\theta) \equiv E(e^{\theta \tilde{L}_1})$. Let $\beta = \sup\{\theta; \tilde{\nu}^{(1)}(\theta) < \infty\}$. Obviously, $\beta \le \alpha_1$. From (1), we have, for $\theta < \beta$,

$$(1 - \varphi((\theta,0))\tilde{\nu}^{(1)}(\theta) = (\varphi^{(1)}((\theta,0)) - \varphi((\theta,0)))E(e^{\theta \tilde{L}_1} 1(\tilde{L}_1 \ge 1, \tilde{L}_2 = 0)) \\ +(\varphi^{(2)}((\theta,0)) - \varphi((\theta,0)))P(\tilde{L}_1 = 0, \tilde{L}_2 \ge 1) + (\varphi^{(0)}((\theta,0)) - \varphi((\theta,0)))P(\tilde{\boldsymbol{L}} = \mathbf{0}). \tag{27}$$

Suppose that $\beta < \alpha_1$. Denote the right-hand side of (27) by $g(\theta)$, and define function $h(w)$ of complex variable w as

$$h(w) = \frac{g(w)}{1 - \varphi((w,0))} \tag{28}$$

as long as it is well defined and finite. If $\varphi((\beta,0)) \ne 1$, then $h(w)$ is analytic for w in a sufficiently small neighborhood of β since $E(e^{\theta \tilde{L}_1} 1(\tilde{L}_1 \ge 1, \tilde{L}_2 = 0))$ is finite for $\theta < \alpha_1$. This implies that $h(w)$ is also analytic in the same neighborhood, which contradicts that $h(\theta) = \tilde{\nu}^{(1)}(\theta)$ diverges for $\theta > \beta$. By a similar reasoning, β can not be 0. Hence, $\varphi((\beta,0)) = 1$ for $\beta > 0$. Thus, we have proved that $\beta < \alpha_1$ implies $\beta = \tilde{\alpha}_1$.

We now consider the case that $\overline{\theta}_2^{(c)} \ge 0$. This and $\beta < \alpha_1 = \theta_1^{(c)}$ imply that $\theta_2^{(c)} > 0$. Then, we can apply (3b) of Proposition 3.2 for $z = e^{\theta_1}$ with $\boldsymbol{u} = \mathbf{1}$ and $\boldsymbol{x} = c_1 e^{-\underline{\theta}_2} + c_2 e^{-\overline{\theta}_2}$ for any θ_1 such that $\beta < \theta_1 < \alpha_1$. Since $\underline{\theta}_2 < \underline{\theta}_2^{(c)} \le \alpha_2$ implies $\overline{d}(z, \boldsymbol{x}) = 0$, we get

$$\limsup_{n\to\infty} e^{\theta n} \nu_n^{(1)} = 0.$$

This contradicts the fact that $\tilde{\nu}^{(1)}(\theta)$ diverges for $\theta > \beta$. Thus, $\beta = \alpha_1$ and we get (4a).

We next consider the case that $\overline{\theta}_2^{(c)} < 0$. Note that $\varphi^{(2)}((\tilde{\alpha}_1, 0)) > 1$ and $\varphi^{(0)}((\tilde{\alpha}_1, 0)) > 1$. Hence, if $\varphi^{(1)}((\tilde{\alpha}_1, 0)) \ge 1$, then the right-hand side of (27) converges to a positive constant as $\theta \uparrow \tilde{\alpha}_1$. From this and the fact that $(1 - \varphi((\theta,0)))/(e^{\tilde{\alpha}_1} - e^{\theta})$ converges to a positive constant as $\theta \uparrow \tilde{\alpha}_1$, it follows that the asymptotic decay of $\nu_n^{(1)}$ is exactly geometric with rate $e^{-\tilde{\alpha}_1}$. Thus, we get (4b1). If $\varphi^{(1)}((\tilde{\alpha}_1, 0)) < 1$, then the right-hand side of (27) may vanish at $\theta = \tilde{\alpha}_1$. If this is the case, the singularity of $h(z)$ of (28) at $z = \tilde{\alpha}_1$ can be removed. Hence, we can only state (4b2). □

Until now, we have been concerned only with the rough decay rates, but we can refine them to obtain the exact asymptotics. We refer to Theorem 5 of [9] for this.

Theorem 4.2 *For the double QBD process satisfying the assumptions (i)–(iii), let*

$$\boldsymbol{\theta}^{\max} \equiv (\theta_1^{\max}, \theta_2^{\max}) = \arg \sup_{(\theta_1,\theta_2)\in\mathcal{D}_0} \{\theta_1\}, \qquad \boldsymbol{\eta}^{\max} \equiv (\eta_1^{\max}, \eta_2^{\max}) = \arg \sup_{(\theta_1,\theta_2)\in\mathcal{D}_0} \{\theta_2\}.$$

If $\alpha_1 > 0$ and $\alpha_2 > 0$, then either one of the following cases holds.

(4c) If $\alpha_1 \ne \theta_1^{\max}$ ($\alpha_2 \ne \eta_2^{\max}$), then the asymptotic decay of $\{\nu_{nk}\}$ ($\{\nu_{kn}\}$) for each fixed $k \in \mathbb{Z}_+$ as $n \to \infty$ is exactly geometric with the rate $e^{-\alpha_1}$ ($e^{-\alpha_2}$).

(4d) If $\alpha_1 = \theta_1^{\max}$ ($\alpha_2 = \eta_2^{\max}$), then we have three cases.

(4d1) If (C1) holds and $\varphi^{(1)}(\boldsymbol{\theta}^{\max}) = 1$ ($\varphi^{(2)}(\boldsymbol{\eta}^{\max}) = 1$), then there exists a positive constant c_k (c'_k) depending on $k \in \mathbb{Z}_+$ such that

$$\lim_{n\to\infty} n^{\frac{1}{2}} e^{n\alpha_1} \nu_{nk} = c_k \qquad (\lim_{n\to\infty} n^{\frac{1}{2}} e^{n\alpha_2} \nu_{kn} = c'_k). \tag{29}$$

(4d2) If (C1) holds and $\varphi^{(1)}(\boldsymbol{\theta}^{\max}) \ne 1$ ($\varphi^{(2)}(\boldsymbol{\eta}^{\max}) \ne 1$), then there exists a positive constant d_k (d'_k) depending on $k \in \mathbb{Z}_+$ such that

$$\lim_{n\to\infty} n^{\frac{3}{2}} e^{n\alpha_1} \nu_{nk} = d_k \qquad (\lim_{n\to\infty} n^{\frac{3}{2}} e^{n\alpha_2} \nu_{kn} = d'_k). \tag{30}$$

(4d3) If (C2) ((C3)) holds, then, for a positive constant e_k (e'_k),

$$\limsup_{n\to\infty} e^{\alpha_1 n}\nu_{nk} = 0, \qquad \lim_{n\to\infty} e^{\alpha_2 n}\nu_{kn} = e_k \tag{31}$$
$$(\lim_{n\to\infty} e^{\alpha_1 n}\nu_{nk} = e'_k, \qquad \limsup_{n\to\infty} e^{\alpha_2 n}\nu_{kn} = 0).$$

Remark 4.6 *$\theta_1^{\max}$ and $\eta_2^{\max}$ may be infinite since $\mathcal{D}_0$ may not be a closed loop. Since (4d) can not occur in this case, the notation $\boldsymbol{\theta}^{\max}$ ($\boldsymbol{\eta}^{\max}$) is used rather than $\boldsymbol{\theta}^{\sup}$ ($\boldsymbol{\eta}^{\sup}$).*

Remark 4.7 *(4d1) and (4d2) are essentially due to Theorem 5 of [9]. However, the latter assumes that the stationary probabilities along the other coordinate decay faster than the decay of entries in the left invariant vector for $A^{(i)}(z)$, where i corresponds with the coordinate of interest. This assumption is hard to check in applications. Theorem 4.2 resolves this difficulty replacing it by condition (C1). Theorem 4.2 also refines the results of [5], which will be detailed in Section 5.*

Remark 4.8 *In case (4d3), $\{\nu_{nk}\}$ for each fixed k decays faster than a geometric sequence if (C2) holds. This is only the case that the exact asymptotics is not known. We conjecture that they have the same exact asymptotics as in (4d1) and (4d2).*

Proof. We first assume that $\alpha_1 \neq \theta_1^{\max}$. Consider case (C3). From Theorem 4.1 and Corollary 4.1, we have $\log\alpha_1 = \theta_1^{(c)} \le \eta_1^{(c)}$, so $\overline{\theta}_2^{(c)} \equiv \max\{\theta_2; (\theta_1^{(c)}, \theta_2) \in \mathcal{D}_0\}$ is greater than $\theta_2^{(c)}$. Hence, vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ with entries $x_\ell = e^{-\ell\overline{\theta}_2^{(c)}}$ and $y_\ell = e^{\ell\theta_2^{(c)}}$ are the left and right eigenvectors of $A_*^{(1)}(e^{\alpha_1})$ in L_1-direction, so $\boldsymbol{x}\boldsymbol{y} < \infty$. Furthermore, $\boldsymbol{\nu}_1\boldsymbol{y} < \infty$ since $\boldsymbol{\nu}_1 \equiv \{\nu_{1n}\}$ has the rough decay rate $e^{-\alpha_2} = e^{-\overline{\theta}_2} < e^{-\theta_2^{(c)}}$. Consequently, all the conditions of Proposition 3.1 are satisfied, so the asymptotic decay of $\{\nu_{nk}\}$ is exactly geometric with rate $e^{-\alpha_1}$ for each fixed k. Due to Proposition 3.2, this in turn implies that the asymptotic decay of $\{\nu_{kn}\}$ and its marginal distribution $\{\nu_n^{(2)}\}$ is exactly geometric with rate $e^{-\alpha_2}$. For case (C2), we have the same results by exchanging the directions. For the case (C1), it is easy to see that the conditions (F1), (F2) and (F3) of Proposition 3.1 are satisfied. Thus, we get (4c).

We next assume that $\alpha_1 = \theta_1^{\max}$. In this case, (C3) is impossible, so we only need to consider cases (C1) and (C2). First assume that (C2) holds, then we must have $(\overline{\eta}_1^{(c)}, \eta_2^{(c)}) = \boldsymbol{\theta}^{\max}$. So, we can apply Proposition 3.1 for the L_2-direction, which implies the asymptotically exact geometric decay of $\{\nu_{kn}\}$ for each fixed k. From this fact and (3c) of Theorem 3.1, we have

$$\overline{d}(e^{\alpha_1}, \boldsymbol{x}) = \limsup_{k\to\infty} \frac{\nu_{1k}}{x_k} = 0,$$

since $x_k = ke^{-\theta_1^{\max}k}$. Hence, (3b) of Proposition 3.2 implies (31). Thus, we get (4d3). We finally consider case (C1) for $\alpha_1 = \theta_1^{\max}$. In this case, we have $\theta_1^{(c)} = \theta_1^{\max}$ and $\alpha_2 = \eta_2^{(c)} > \theta_2^{(c)} = \theta_2^{\max}$. Hence, we can apply the bridge path and jitter cases of Theorem 5 of [9], which lead to (4d2) and (4d1), respectively. □

We next give the exact asymptotics version of Corollary 4.3.

Corollary 4.4 *For the double QBD process satisfying the assumptions (i)–(iii), assume that $\alpha_1 > 0$ and $\alpha_2 > 0$. Then, if $\overline{\theta}_2^{(c)} < 0$ and if $\varphi^{(1)}(\tilde{\alpha}_1) \ge 1$, then the asymptotic decay of $\nu_n^{(1)}$ is exactly geometric with rate $e^{-\tilde{\alpha}_1}$, where $\tilde{\alpha}_1$ is defined in Corollary 4.3. If $\overline{\theta}_2^{(c)} \ge 0$, we have the following cases.*

(4e) If $\alpha_1 \neq \theta_1^{\max}$, then we have three cases.

(4e1) If $\overline{\theta}_2^{(c)} > 0$ and $\theta_2^{(c)} \neq 0$, then the asymptotic decay of $\{\nu_n^{(1)}\}$ is exactly geometric with the rate $e^{-\alpha_1}$.

(4e2) If $\overline{\theta}_2^{(c)} > 0$ and $\theta_2^{(c)} = 0$, then, for some $c \ge 0$,

$$\lim_{n\to\infty} e^{\alpha_1 n}\nu_n^{(1)} = c. \tag{32}$$

(4e3) If $\overline{\theta}_2^{(c)} = 0$, then, for some $d > 0$,

$$\lim_{n\to\infty} n^{-1}e^{\alpha_1 n}\nu_n^{(1)} = d. \tag{33}$$

(4f) If $\alpha_1 = \theta_1^{\max}$, we have another three cases.

(4f1) If $\overline{\theta}_2^{(c)} > 0$, then $\nu_n^{(1)}$ has the same exact asymptotics as ν_{nk} for each fixed k except a multiplicative constant.

(4f2) If $\overline{\theta}_2^{(c)} = 0$ and $\varphi^{(1)}((\alpha_1, 0)) = 1$, then the asymptotic decay of $\{\nu_n^{(1)}\}$ is exactly geometric with the rate $e^{-\alpha_1}$.

(4f3) If $\overline{\theta}_2^{(c)} = 0$ and $\varphi^{(1)}((\alpha_1, 0)) \neq 1$, then we have (32) for a $c \geq 0$.

Proof. For the case that $\overline{\theta}_2^{(c)} < 0$ and $\varphi^{(1)}(\tilde{\alpha}_1) \geq 1$, we have already proved the asymptotically exact geometric decay in the proof of Corollary 4.3. So, we assume that $\overline{\theta}_2^{(c)} \geq 0$. This and Corollary 4.3 imply that the rough decay rate of $\nu_n^{(1)}$ is α_1. To prove (4e), assume that $\alpha_1 \neq \theta_1^{\max}$. In this case, we always have $\alpha_1 \leq \theta_1^{(c)}$ and $\varphi(\boldsymbol{\theta}^{(c)}) = \varphi^{(1)}(\boldsymbol{\theta}^{(c)}) = 1$. If $\alpha_1 \neq \theta_1^{(c)}$, then case (C2) occurs, and $\nu_n^{(1)}$ has the exactly geometric decay with rate α_1 by Proposition 3.2. Note that we must have $\theta_2^{(c)} > 0$ for this case, and (4e1) is obtained for $\alpha_1 \neq \theta_1^{(c)}$. Hence, we only need to consider the case where $\alpha_1 = \theta_1^{(c)}$. Assume that $\overline{\theta}_2^{(c)} > 0$, and consider two cases $\theta_2^{(c)} \neq 0$ and $\theta_2^{(c)} = 0$ separately. If $\theta_2^{(c)} \neq 0$ holds, then $1 - \varphi((\alpha_1, 0))$ and $\varphi^{(1)}((\alpha_1, 0)) - 1$ have the same sign. Hence, from (27), it can be seen that $\nu_n^{(1)}$ and ν_{n0} have the same exact asymptotics, so we have (4e1) by (4c) of Theorem 4.2. If $\theta_2^{(c)} = 0$, then $\varphi((\alpha_1, 0)) = \varphi((\alpha_1, 0)) = 1$. So, letting $\theta \uparrow \alpha_1$ in (27), we can see that its right-hand side converges to a nonnegative constant. Hence, we get (4e2). We next assume that $\overline{\theta}_2^{(c)} = 0$, which excludes case (C2). So, $\tilde{\alpha}_1 = \alpha_1 = \theta_1^{(c)}$ by Corollary 4.1, and and $\varphi^{(1)}((\alpha_1, 0)) > 1$ by Theorem 3.1. Since ν_{n0} has the asymptotically exact decay rate α_1, $(e^{\alpha_1} - e^{\theta})E(e^{\theta \tilde{L}_1} 1(\tilde{L}_1 \geq 1, \tilde{L}_2 = 0))$ converges to a positive constant as $\theta \uparrow \alpha_1$. Furthermore, $\varphi((\alpha_1, 0)) = 1$ implies that $(1 - \varphi((\theta, 0)))/(e^{\alpha_1} - e^{\theta})$ converges to a positive constant as $\theta \uparrow \alpha_1$. Hence, multiplying both sides of (27) with $(e^{\alpha_1} - e^{\theta})$, we have, for some positive a,

$$\lim_{\theta \uparrow \alpha_1} (e^{\alpha_1} - e^{\theta})^2 E(e^{\theta \tilde{L}_1}) = a.$$

This yields (33), and (4e3) is obtained. To prove (4f), assume that $\alpha_1 = \theta_1^{\max}$. If $\overline{\theta}_2^{(c)} > 0$, then $\alpha_1 = \theta_1^{\max} > \tilde{\alpha}_1$, which implies that $\varphi((\alpha_1, 0)) > 1$ and $\varphi^{(1)}((\alpha_1, 0)) \leq 1$. But, $\varphi^{(1)}((\alpha_1, 0)) = 1$ is impossible since this and (27) with $\theta \uparrow \alpha_1$ imply that $\tilde{\nu}^{(1)}(\alpha_1) < \infty$ and therefore the rough decay rate of $\nu_n^{(1)}$ is less than α_1. So, $\varphi^{(1)}((\alpha_1, 0)) < 1$, and we have (4f1) from (27). If $\overline{\theta}_2^{(c)} = 0$, then $\alpha_1 = \theta_1^{\max} = \tilde{\alpha}_1$, so $\varphi((\alpha_1, 0)) = 1$. We need to consider (27) for the two cases according to whether $\varphi^{(1)}((\alpha_1, 0)) = 1$ or not. Note that, if $\varphi^{(1)}((\alpha_1, 0)) \neq 1$, then $\varphi^{(1)}((\alpha_1, 0)) < 1$ by $\alpha_1 = \theta_1^{\max}$. If $\varphi^{(1)}((\alpha_1, 0)) = 1$, then, by (4d1) of Theorem 4.2, where (C2) and (C3) are impossible by $\alpha_2 > 0$, we have

$$\lim_{\theta \uparrow \alpha_1} (\varphi^{(1)}((\alpha_1, 0)) - \varphi((\alpha_1, 0))) E(e^{\theta \tilde{L}_1} 1(\tilde{L}_1 \geq 1, \tilde{L}_1 = 0)) = 0.$$

Hence, (27) yields the asymptotically exact geometric decay. On the other hand, if $\varphi^{(1)}((\alpha_1, 0)) < 1$, then, by (4d2) of Theorem 4.2

$$\lim_{\theta \uparrow \alpha_1} E(e^{\theta \tilde{L}_1} 1(\tilde{L}_1 \geq 1, \tilde{L}_1 = 0)) < \infty.$$

Hence, the right-hand side of (27) converges to a constant as $\theta \uparrow \alpha_1$. Since this constant may be zero because of $\varphi^{(1)}((\alpha_1, 0)) < 1$, we can only state that the decay is not slower than the geometric decay with rate $e^{-\alpha_1}$. Thus, we get (4f2) and (4f3), and complete the proof. □

In conclusion, we have a complete answer to the rough decay rate problem for each fixed phase. Similarly, the exact asymptotics is obtained except for (4d3). For the marginal distributions, the rough and exact asymptotics are also obtained except for some cases. Thus, we have a fairly good view for the rough and exact asymptotics for the double QBD process in the sense that α_1 and α_2 are graphically identified through cases (C1), (C2) and (C3). However, this does not mean that the decay rates are easily obtained in applications. Therefore, we still need to employ some effort to identify the decay rates. Such examples are demonstrated in Section 6.

5. 0-partially homogeneous chain. In the previous sections, we have considered the decay rates only for the coordinate directions and the marginal stationary distributions. For theoretical reasons, one may be interested in the decay rate for an arbitrarily given direction. In [5], this problem is studied for much more general processes. In particular, the exact asymptotics are considered for the 0-partially homogeneous chain there. In this section, we revisit the decay rate problem through the 0-partially homogenous chain. We here refine the decay rates of [5] to be more explicit. We also remark on some technical issues in the results of [5] that they require stronger conditions and the marginal stationary distributions are not considered.

Let us formally introduce the 0-partially homogeneous chain of [5]. Let S be either $\mathbb{R}^2$ or $\mathbb{Z}^2$, where $\mathbb{Z}$ is the set of all integers, and let

$$\begin{aligned} S_+ &= \{(v_1, v_2) \in S; v_1, v_2 \geq 0\}, & S_+^{\mathrm{o}} &= \{(v_1, v_2) \in S; v_1, v_2 > 0\}, \\ \partial S_+^{(1)} &= \{(y, 0) \in S_+; y > 0\}, & \partial S_+^{(2)} &= \{(0, y) \in S_+; y > 0\}, \\ S^{(1)} &= \{(v_1, v_2) \in S; v_2 \geq 0\}, & S^{(2)} &= \{(v_1, v_2) \in S; v_1 \geq 0\}, \end{aligned}$$

and let $\mathcal{B}(S_+)$ be the Borel field on S_+. Let $\boldsymbol{X}$, $\boldsymbol{X}^{(0)}$, $\boldsymbol{X}^{(1)}$ and $\boldsymbol{X}^{(2)}$ be random vectors taking values in $S, S_+, S^{(1)}$ and $S^{(2)}$, respectively.

A discrete-time Markov process $\{\boldsymbol{Y}_n\}$ is said to be a two dimensional 0-partially homogenous chain if it takes values in S_+ and if its transition kernel $K(\boldsymbol{v}, B) \equiv P(\boldsymbol{Y}_{n+1} \in B | \boldsymbol{Y}_n = \boldsymbol{v})$ has the following form:

$$K(\boldsymbol{v}, B) = \begin{cases} P(\boldsymbol{X} \in B - \boldsymbol{v}), & \boldsymbol{v} \in S_+^{\mathrm{o}}, B \in \mathcal{B}(S_+^{\mathrm{o}}), \\ P(\boldsymbol{X}^{(0)} \in B), & \boldsymbol{v} = \boldsymbol{0}, B \in \mathcal{B}(S_+), \\ P(\boldsymbol{X}^{(1)} \in B - \boldsymbol{v}), & \boldsymbol{v} \in \partial S^{(1)}, B \subset S_+ \setminus \partial S_+^{(2)}, B \in \mathcal{B}(S_+), \\ P(\boldsymbol{X}^{(2)} \in B - \boldsymbol{v}), & \boldsymbol{v} \in \partial S^{(2)}, B \subset S_+ \setminus \partial S_+^{(1)}, B \in \mathcal{B}(S_+), \end{cases}$$

where the other $K(\boldsymbol{v}, B)$ can be arbitrary as long as $K(\boldsymbol{v}, S) = 1$. Note that $K(\boldsymbol{v}, B)$ may depend on $\boldsymbol{v}$ for $(\boldsymbol{v}, B) \in \partial S_+^{(1)} \times \partial S_+^{(2)}$ or $(\boldsymbol{v}, B) \in \partial S_+^{(2)} \times \partial S_+^{(1)}$.

Clearly, $\{\boldsymbol{Y}_n\}$ is the double QBD process if $S = \mathbb{Z}^2$ and it is skip free in all directions. Similarly to $\{\boldsymbol{L}_n^{(1)}\}$ and $\{\boldsymbol{L}_n^{(2)}\}$, we define additive processes removing either the boundary $\partial S_+^{(2)}$ or $\partial S_+^{(1)}$, which are denoted by $\{\boldsymbol{Y}_n^{(1)}\}$ and $\{\boldsymbol{Y}_n^{(2)}\}$, respectively. They are Markov additive processes. We also need a random walk $\{\boldsymbol{Z}_n\}$, where $\boldsymbol{Z}_n = \boldsymbol{X}_1 + \ldots + \boldsymbol{X}_n$ for independently and identically distributed random vectors $\boldsymbol{X}_\ell$ which have the same distribution as $\boldsymbol{X}$.

5.1 Borovkov and Mogul'skii's solution. We briefly summarize the decay rates obtained in [5] for the 0-partially homogenous chain $\{\boldsymbol{Y}_n\}$. For simplicity, we are only concerned with the case of $S = \mathbb{Z}^2$. Results for $S = \mathbb{R}^2$ are essentially parallel although conditions are more complicated.

The approach of [5] uses the large deviations technique, and requires the following conditions in addition to the stability condition (see (R_1) and (R_2) of [5]).

(G1) For each $\boldsymbol{j} \in \mathbb{Z}^2$ such that $P(\boldsymbol{X} = \boldsymbol{j}) > 0$, the group generated by $\{\boldsymbol{k} \in \mathbb{Z}^2; P(\boldsymbol{X} = \boldsymbol{k} + \boldsymbol{j}) > 0\}$ agrees with $\mathbb{Z}^2$.

(G2) The local processes $\{\boldsymbol{Y}_n^{(1)}\}$ and $\{\boldsymbol{Y}_n^{(2)}\}$ are irreducible.

(G3) $m_i \equiv E(X^{(i)}) \neq 0$ for $i = 1, 2$.

The same set of conditions is used in [17], which also uses the large deviations approach. In [5], the finiteness of moment generating functions on $\boldsymbol{X}$ is assumed together with some other assumptions, but this is required to have geometric decay.

Among these conditions, we only need (G2) for the double QBD process. That is, (G2) is identical with (ii), and (G1) and (G3) are not needed. The latter two conditions may exclude theoretically interesting cases (see, e.g., Section 28 of [3]). Furthermore, condition (ii) can be removed, which will be discussed in Section 7. Thus, our approach in the previous sections can be used under much weaker assumptions although it is only applicable for the double QBD process.

Similarly to the double QBD process, we use the following notation.

$$\varphi(\boldsymbol{\theta}) = E(e^{\boldsymbol{\theta X}}), \quad \boldsymbol{\theta} \in \mathbb{R}^2.$$

Then, the moment condition (M^+) assumed in [5] can be written as

(M1) $\overline{\varphi}(\boldsymbol{\theta}) \equiv \sup_{\boldsymbol{v}\in\mathbb{R}^2_+} \int_{\mathbb{R}^2_+} e^{\boldsymbol{\theta u}} K(\boldsymbol{v}, d\boldsymbol{u}) < \infty$ for some $\boldsymbol{\theta} > \mathbf{0}$, and $\overline{\varphi}(\boldsymbol{\theta}) < \infty$ for all $\boldsymbol{\theta}$ such that $\varphi(\boldsymbol{\theta}) < \infty$.

Obviously, this condition is automatically satisfied for the double QBD process.

We now present the rough decay rates of [5] for $S = \mathbb{Z}^2$. For this, we need some notation. For the Markov additive process $\{\boldsymbol{Y}^{(1)}_n\}$ with $\boldsymbol{Y}^{(1)}_0 = \mathbf{0}$, we inductively define, for $n = 1, 2, \ldots$,

$$\tau^{(1)}_n = \inf\{n \geq \tau^{(1)}_{n-1} + 1; Y^{(1)}_{2n} = 0\}, \qquad \zeta^{(1)}_n = Y^{(1)}_{1\tau^{(1)}_n},$$

where $\tau^{(1)}_0 = 0$. That is, $\tau^{(1)}_n$ is the n-th time when the Markov component $Y^{(1)}_{2n}$ returns to 0. Then, let $\psi^{(1)}(t) = E(e^{\zeta^{(1)}_1 t}; \tau^{(1)}_1 < \infty | (\boldsymbol{Y}^{(1)}_0) = \mathbf{0})$ and

$$\delta^{(1)} = \sup\{t; \psi^{(1)}(t) \leq 1\}.$$

We similarly define $\psi^{(2)}$ and $\delta^{(2)}$. Let

$$D(\boldsymbol{\beta}) = \sup\{\boldsymbol{\beta\theta}^{\mathrm{T}}; \varphi(\boldsymbol{\theta}) \leq 1\}, \qquad \boldsymbol{\beta} \in \mathbb{Z}^2$$
$$D_i(\mathbf{0}, \boldsymbol{\beta}) = \inf\{\delta^{(i)} t_i + D(\boldsymbol{\beta} - \boldsymbol{t}); \boldsymbol{t} = (t_1, t_2) \in \partial S^{(i)}_+\}, \qquad \boldsymbol{\beta} \geq \mathbf{0},\ i = 1, 2,$$
$$D(\mathbf{0}, \boldsymbol{\beta}) = \min(D(\boldsymbol{\beta}), D_1(\mathbf{0}, \boldsymbol{\beta}), D_2(\mathbf{0}, \boldsymbol{\beta})), \qquad \boldsymbol{\beta} \geq \mathbf{0},$$

where the original definition of $D(\boldsymbol{\beta})$, defined in [5] as

$$D(\boldsymbol{\beta}) = \inf_{t>0} t\Lambda(\frac{1}{t}\boldsymbol{\beta}),$$

using Frenchel-Legendre transform $\Lambda(\boldsymbol{\beta}) = \sup_{\boldsymbol{\theta}\in\mathbb{R}^2}\{\boldsymbol{\beta\theta} - \log \varphi(\boldsymbol{\theta})\}$ is different. In [2] (see also [4]), it is proved that

$$\lim_{\ell\to\infty} \frac{1}{\ell} \log H(\ell\boldsymbol{\beta}) = -D(\boldsymbol{\beta}), \qquad \boldsymbol{\beta} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\},$$

where $H(\boldsymbol{n}) = \sum_{\ell=0}^{\infty} P(\boldsymbol{Y}_\ell \in \boldsymbol{n})$. Since H is the two dimensional renewal function whose increments have moment generating function φ, it is intuitively clear that the rough decay rate $D(\boldsymbol{\beta})$ must be the one which we have defined. Note that this equivalence generally holds true (see Theorem 13.5 of [30]).

In large deviations terminology, $D(\boldsymbol{\beta})$ represents the minimum cost that the random walk $\{\boldsymbol{Z}_n\}$ moves from $\mathbf{0}$ to $\boldsymbol{\beta}$ under a fluid scaling for any $\boldsymbol{\beta} \neq \mathbf{0}$. If $\boldsymbol{\beta} \geq \mathbf{0}$, then it is also the minimum cost for $\{\boldsymbol{Y}_n\}$ when it only moves inside the quadrant. Similarly, $\delta^{(i)}$ is the minimum unit cost when $\{\boldsymbol{Y}_n\}$ moves on the boundary $\partial S^{(i)}_+$, so $D_i(\mathbf{0}, \boldsymbol{\beta})$ is the minimum cost for moving from $\mathbf{0}$ on the boundary $\partial S^{(i)}_+$ then goes to $\boldsymbol{\beta} \geq \mathbf{0}$. Thus, $D(\mathbf{0}, \boldsymbol{\beta})$ is the minimum cost moving from $\mathbf{0}$ to $\boldsymbol{\beta} \geq \mathbf{0}$, and gives the rough decay rate of the stationary distribution $\boldsymbol{\nu}$ in the direction determined by $\boldsymbol{\beta}$.

Proposition 5.1 (Theorem 3.1 of Borovkov and Mogul'skii [5]) *For the 0-partially homogeneous chain $\{\boldsymbol{Y}_n\}$, assume that it has the stationary distribution $\boldsymbol{\nu} \equiv \{\nu_{\boldsymbol{i}}; \boldsymbol{i} \in S_+\}$ and satisfies (G1), (G2), (G3) and (M1). Then,*

$$\lim_{\ell\to\infty} \frac{1}{\ell} \log \nu_{\ell\boldsymbol{\beta}} = -D(\mathbf{0}, \boldsymbol{\beta}), \qquad \boldsymbol{\beta} \in \mathbb{Z}^2_+ \setminus \{\mathbf{0}\}. \tag{34}$$

Remark 5.1 *From (34), we must have $D(\mathbf{0}, t\boldsymbol{\beta}) = tD(\mathbf{0}, \boldsymbol{\beta})$ for $t \geq 0$, which is obtained in Theorem 1.4 of [5]. It should be noted that (34) is the rough decay rate on the ray with direction $\boldsymbol{\beta}$, and the case of $\boldsymbol{\beta} = \mathbf{e}_i$ does not imply the rough decay rate of the marginal stationary distribution.*

This result is much more general than Theorem 4.1 except for the extra conditions (G1) and (G3), but nothing tells about the rough decay rate of the marginal distributions. From the viewpoint of applications, a big problem is that the rough decay rate $D(\mathbf{0}, \boldsymbol{\beta})$ is not easy to compute. Borovkov and Mogul'skii [5] consider a slightly different characterization for $\boldsymbol{\beta} = \mathbf{e}_i$ for $i = 1, 2$, where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$. Let

$$g^{(1)}(u) = \min_{v\geq 0}\{uv + D((1, -v))\}, \qquad g^{(2)}(u) = \min_{v\geq 0}\{uv + D((-v, 1))\}. \tag{35}$$

Obviously, $D_2(\mathbf{0}, \mathbf{e}_1) = g^{(1)}(\delta^{(2)})$ and $D_1(\mathbf{0}, \mathbf{e}_2) = g^{(2)}(\delta^{(1)})$. Then, it is shown in [5] that

$$\lambda^{(1)} = \min(\delta^{(1)}, g^{(1)}(\lambda^{(2)})), \qquad \lambda^{(2)} = \min(\delta^{(2)}, g^{(2)}(\lambda^{(1)})) \tag{36}$$

has a unique positive solution $(\lambda^{(1)}, \lambda^{(2)})$, and $\lambda^{(i)} = D(\mathbf{0}, \mathbf{e}_i)$ for $i = 1, 2$. This fixed point idea is interesting, but (36) does not help to get $\lambda^{(1)}$ and $\lambda^{(2)}$ from the primitive modeling data since $g^{(1)}(\lambda^{(2)})$ and $g^{(2)}(\lambda^{(1)})$ are not explicitly given. As we shall see in the next section, they can be obtained in closed form if a slightly stronger moment condition is assumed (see (41) and (42)). Then, $D(\mathbf{0}, \boldsymbol{\beta})$ can be obtained in much nicer form.

We next consider the exact asymptotics of [5]. For this case, the results corresponding to Proposition 5.1 are obtained for $\nu_{n\boldsymbol{\beta}}$ as $n \to \infty$ for each $\boldsymbol{\beta} \in \mathbb{Z}_+^2$ further assuming a 1-arithmetic type condition corresponding to (iii) of Section 2. We here only present their results for the coordinate directions.

Proposition 5.2 (The first part of Theorem 3.3 of [5]) *Under the aforementioned condition in addition to those of Proposition 5.1, let $\chi^{(1)} = g^{(1)}(\lambda^{(2)})$ and $\chi^{(2)} = g^{(2)}(\lambda^{(1)})$, and suppose either one of the following conditions (A1) and (B1) holds.*

(A1) $\delta^{(1)} < \min(\chi^{(1)}, \theta_1^{\max})$ *or* $\delta^{(1)} > \chi^{(1)}, \delta^{(2)} < \min(\chi^{(2)}, \eta_2^{\max})$,

(B1) $\lambda^{(2)} > \theta_2^{\max}, \psi^{(1)}(\delta^{(1)}) < 1$ *or* $\lambda^{(2)} < \theta_2^{\max}, \lambda^{(1)} > \eta_1^{\max}, \psi^{(2)}(\delta^{(2)}) < 1$.

Then, (A1) implies that, for a positive constant c_1,

$$\lim_{n\to\infty} e^{\lambda^{(1)} n} \nu_{n0} = c_1, \tag{37}$$

and (B1) implies that, for a positive constant c'_1,

$$\lim_{n\to\infty} n^{\frac{3}{2}} e^{\lambda^{(1)} n} \nu_{n0} = c'_1. \tag{38}$$

A corresponding result is also obtained for the direction $\mathbf{e}_2$.

Remark 5.2 *In Theorem 3.3 of [5], $\overline{\nu}_{n0} \equiv \sum_{\ell=n}^{\infty} \nu_{\ell 0}$ is used instead of ν_{n0}, but this does not change the exact asymptotics except for a constant multiplier since ν_{n0} has the rough decay rate less than 1.*

Conditions (A1) and (B1) are not so informative although it is remarked in [5] that they cover all the cases except for parameter sets of Lebesque measure 0. As we remarked, we can convert them to more explicit conditions if we assume the moment condition (M2) given in the next section. For the double QBD process, (M2) is always satisfied, and $\delta^{(1)} = \theta_1^{(c)}$ and $\delta^{(2)} = \eta_2^{(c)}$ by Corollary 5.2 in the next section. From these facts, it can be seen that (A1) is equivalent to that (C1) with $\lambda^{(1)} \neq \theta_1^{\max}$, (C3) or (C2) with $\overline{\lambda}_1^{(2)} \neq \theta_1^{(c)}$, where $\overline{\lambda}_1^{(2)} = \max\{\theta_1; \varphi((\theta_1, \lambda^{(2)})) = 1\}$. Since (C3) or (C2) with $\overline{\lambda}_1^{(2)} \neq \theta_1^{(c)}$ implies $\lambda^{(1)} \neq \theta_1^{\max}$, (A1) indeed implies case (4c) of Theorem 4.2, but (A1) excludes (C2) with $\overline{\lambda}_1^{(2)} = \theta_1^{(c)} < \theta_1^{\max}$ which is in (4c). Thus, (A1) is a subset of (4c). On the other hand, it is not hard to see that (B1) is equivalent to (4d2) of Theorem 4.2. The other cases, that is, (4d1) and (4d3) are dropped in Proposition 5.2.

Thus, there remain the following problems in [5].

(P1) The extra assumptions (G1) and (G3) are needed.

(P2) There are missing cases in (A1) and (B1) of Proposition 5.2,

(P3) The exact asymptotics is not studied for ν_{nk} and ν_{kn} for each fixed $k \geq 1$.

(P4) Nothing is considered for the rough and exact asymptotics of the marginal stationary distributions.

In particular, Borovkov and Mogul'skii [5] remark that their approach can not be used for the marginal distributions. As we have shown in Section 4, we have resolved the above problems for the double QBD process. Specifically, (P1), (P2) and (P3) are well resolved by Theorems 4.1 and 4.2. For the marginal distributions, Corollaries 4.3 and 4.4 solve this decay rate problem in some extent.

5.2 Computing the decay rates. Proposition 5.1 is a general result, but it is hard to use for applications since we have to solve optimization problems to get $D(\mathbf{0}, \boldsymbol{\beta})$. Let us consider a computationally tractable form for this rough decay rate. Theorem 4.1 suggests that it would be uniquely determined by a point on the curve the $\varphi(\boldsymbol{\theta}) = 1$. This motivates us to define the following condition in addition to (M1).

(M2) For some $\epsilon > 0$, $\{\boldsymbol{\theta} \in \mathbb{R}_+^2; \varphi(\boldsymbol{\theta}) \le 1+\epsilon\}$ is a simply connected closed set, where $\mathbb{R}_+$ is the set of all nonnegative numbers.

This condition is satisfied by the double QBD process. Furthermore, (G1) implies that $\mathcal{D}_0 \equiv \{\boldsymbol{\theta} \in \mathbb{R}^2; \varphi(\boldsymbol{\theta}) = 1\}$ is a bounded closed loop. Before using (M2), we note the following fact, which is obtained in Lemma 4.4 of [5].

Lemma 5.1 *Under condition (M1), $\delta^{(i)} \le D(\mathbf{e}_i)$ for i=1,2.*

We now assume condition (M2), and solve the optimization problems in $D_i(\mathbf{0},\boldsymbol{\beta})$ for $i=1,2$. From (M2), $\varphi(\boldsymbol{\theta})$ is continuously differentiable in each component of $\boldsymbol{\theta}$ in the domain $\{\boldsymbol{\theta} \in \mathbb{R}^2; \varphi(\boldsymbol{\theta}) < 1+\epsilon\}$. Hence, $D(\boldsymbol{\beta})$ is attained on the closed loop $\mathcal{D}_0$, which is the boundary of a convex set. Let

$$\theta_1^{\max} = \sup\{\theta_1; (\theta_1,\theta_2) \in \mathcal{D}_0\}, \qquad \eta_2^{\max} = \sup\{\eta_2; (\eta_1,\eta_2) \in \mathcal{D}_0\},$$

then, it is not hard to see that

$$D(\mathbf{e}_1) = \theta_1^{\max}, \qquad D(\mathbf{e}_2) = \eta_2^{\max}, \qquad D_i(\mathbf{0},\mathbf{e}_i) = \delta^{(i)}, \qquad i=1,2, \tag{39}$$

which are also intuitively clear. The following lemma is a key for our arguments, and proved in Appendix E.

Lemma 5.2 *Assume condition (M2). For $i=1,2$, define the function $h^{(i)}(u,\boldsymbol{\beta})$ as*

$$h^{(i)}(u,\boldsymbol{\beta}) = \min_{v\ge 0}(uv + D(\boldsymbol{\beta} - v\mathbf{e}_i)), \qquad 0 \le u \le D(\mathbf{e}_i), \boldsymbol{\beta} \in \mathbb{Z}_+^2,$$

then we have

$$h^{(i)}(u,\boldsymbol{\beta}) = \begin{cases} \boldsymbol{\beta}\overline{\boldsymbol{u}}^{(i)}, & u \le \theta_i^{\max}(\boldsymbol{\beta}), \\ D(\boldsymbol{\beta}), & u > \theta_i^{\max}(\boldsymbol{\beta}), \end{cases} \qquad i=1,2, \tag{40}$$

where $\overline{\boldsymbol{u}}^{(1)} = (u,\overline{u}_2)^{\mathrm{T}}$ ($\overline{\boldsymbol{u}}^{(2)} = (\overline{u}_1,u)^{\mathrm{T}}$) for $\overline{u}_2 = \max\{\theta_2; (u,\theta_2) \in \mathcal{D}_0\}$ ($\overline{u}_1 = \max\{\theta_2; (\theta_1,u) \in \mathcal{D}_0\}$) (respectively).

By this lemma, we can compute $g^{(i)}(u)$ of (35) under condition (M2). In particular,

$$\chi^{(1)} = g^{(1)}(\lambda^{(2)}) = h^{(1)}(\lambda^{(2)},\mathbf{e}_1) = \begin{cases} \overline{\lambda}_1^{(2)}, & \lambda^{(2)} \le \theta_1^{\max}, \\ \theta_1^{\max}, & \lambda^{(2)} > \theta_1^{\max}, \end{cases} \tag{41}$$

$$\chi^{(2)} = g^{(2)}(\lambda^{(1)}) = h^{(2)}(\lambda^{(1)},\mathbf{e}_2) = \begin{cases} \overline{\lambda}_2^{(1)}, & \lambda^{(1)} \le \eta_2^{\max}, \\ \eta_2^{\max}, & \lambda^{(1)} > \eta_2^{\max}. \end{cases} \tag{42}$$

We next compute the decay rate $D(\mathbf{0},\boldsymbol{\beta})$. For $u = \delta^{(i)}$, we denote $\overline{\boldsymbol{u}}^{(i)}$ by $\overline{\boldsymbol{\delta}}^{(i)}$. Then,

$$D_i(\mathbf{0},\boldsymbol{\beta}) = h^{(i)}(\delta^{(i)},\boldsymbol{\beta}), \qquad i=1,2.$$

This and Lemma 5.2 immediately lead to the next theorem since $D(\boldsymbol{\beta}) = \boldsymbol{\beta}\boldsymbol{\theta}^{\max} \ge \boldsymbol{\beta}\overline{\boldsymbol{\delta}}^{(i)}$.

Theorem 5.1 *Under condition (M2) in addition to those of Proposition 5.1, we have, for each $\boldsymbol{\beta} \in \mathbb{Z}_+^2$,*

$$D_i(\mathbf{0},\boldsymbol{\beta}) = \begin{cases} \boldsymbol{\beta}\overline{\boldsymbol{\delta}}^{(i)}, & \delta^{(i)} \le \theta_i^{\max}(\boldsymbol{\beta}), \\ D(\boldsymbol{\beta}), & \delta^{(i)} > \theta_i^{\max}(\boldsymbol{\beta}), \end{cases} \qquad i=1,2, \tag{43}$$

where $\boldsymbol{\theta}^{\max}(\boldsymbol{\beta}) = \arg\max_{\boldsymbol{\theta}\in\mathcal{D}_0} \boldsymbol{\beta}\boldsymbol{\theta}^{\mathrm{T}}$, and the rough decay rate of (34) is given by

$$D(\mathbf{0},\boldsymbol{\beta}) = \begin{cases} \min(\boldsymbol{\beta}\overline{\boldsymbol{\delta}}^{(1)}, \boldsymbol{\beta}\overline{\boldsymbol{\delta}}^{(2)}), & (\delta^{(1)},\delta^{(2)}) \le \boldsymbol{\theta}^{\max}(\boldsymbol{\beta}), \\ \boldsymbol{\beta}\overline{\boldsymbol{\delta}}^{(1)}, & \delta^{(1)} \le \theta_1^{\max}(\boldsymbol{\beta}), \theta_2^{\max}(\boldsymbol{\beta}) < \delta^{(2)}, \\ \boldsymbol{\beta}\overline{\boldsymbol{\delta}}^{(2)}, & \theta_1^{\max}(\boldsymbol{\beta}) < \delta^{(1)}, \delta^{(2)} \le \theta_2^{\max}(\boldsymbol{\beta}), \\ D(\boldsymbol{\beta}), & \boldsymbol{\theta}^{\max}(\boldsymbol{\beta}) < (\delta^{(1)},\delta^{(2)}). \end{cases} \tag{44}$$

Remark 5.3 *In the proof of Lemma 5.2, we do not use the fact that $\boldsymbol{\beta} \in \mathbb{Z}_+^2$ as well as any lattice property. So, this theorem is also valid for $S = \mathbb{R}^2$ and $\boldsymbol{\beta} \in \mathbb{R}_+^2$.*

Remark 5.4 *If we view the points $\overline{\boldsymbol{\delta}}^{(1)}$ and $\overline{\boldsymbol{\delta}}^{(2)}$ as $\overline{\boldsymbol{\theta}}^{(c)}$ and $\overline{\boldsymbol{\eta}}^{(c)}$, respectively, for the double QBD process, then we can have a similar geometric interpretation to Remark 4.3 on how they determine the decay rate although things are complicated due to the direction $\boldsymbol{\beta}$. For example, if $\delta^{(1)} > \theta_1^{\max}$ or $\delta^{(2)} > \eta_2^{\max}$, then it is impossible to have $(\delta^{(1)}, \delta^{(2)}) \le \boldsymbol{\theta}^{\max}(\boldsymbol{\beta})$ for all $\boldsymbol{\beta}$. Hence, there are three regions for the decay rates, $\boldsymbol{\beta}\overline{\boldsymbol{\delta}}^{(1)}$, $D(\boldsymbol{\beta})$ and $\boldsymbol{\beta}\overline{\boldsymbol{\delta}}^{(2)}$, which are typically arising in applications.*

Remark 5.5 *If (M2) does not hold, we have to consider the convex set $\{\boldsymbol{\theta} \ge 0; \varphi(\boldsymbol{\theta}) \le 1\}$ instead of $\mathcal{D}_0$. From the geometric interpretation of (43), we can expect to have a similar solution, but the proof does not work. This case is left for future work.*

The following corollary is immediate from Theorem 5.1 for $\boldsymbol{\beta} = \mathbf{e}_i$ since $\delta^{(i)} \le \theta_i^{\max}(\mathbf{e}_i) = \gamma_i$ always holds.

Corollary 5.1 *Under the assumptions of Theorem 5.1, we have, for $i = 1, 2$,*

$$\lambda_i = D(\mathbf{0}, \mathbf{e}_i) = \begin{cases} \min(\delta^{(i)}, \overline{\delta}_i^{(3-i)}), & \delta^{(3-i)} \le \theta_{3-i}^{\max}(\mathbf{e}_i), \\ \delta^{(i)}, & \delta^{(3-i)} > \theta_{3-i}^{\max}(\mathbf{e}_i). \end{cases} \tag{45}$$

Remark 5.6 *(45) with Theorem 5.1 tells us how the optimal path is chosen for the decay rate in the coordinate directions. For example, consider the case that $i = 1$. Note that $D_1(\mathbf{0}, \mathbf{e}_1) = \delta^{(1)}$ and $D_2(\mathbf{0}, \mathbf{e}_1) = \overline{\delta}_1^{(2)}$. When $D(\mathbf{0}, \mathbf{e}_1) = D_1(\mathbf{0}, \mathbf{e}_1)$, the optimal path is said to be jitter and bridge, respectively, if $\delta^{(1)} < D(\mathbf{e}_1)$ and $\delta^{(1)} = D(\mathbf{e}_1)$. When $D(\mathbf{0}, \mathbf{e}_1) = D_2(\mathbf{0}, \mathbf{e}_1)$, the optimal path is said to be cascade in the literature (see, e.g., [8]). Hence, $\lambda_1 = \delta^{(1)}$ occurs only if the optimal path is jitter or bridge while $\lambda_1 = \overline{\delta}_1^{(2)}$ occurs only if the optimal path is cascade. The jitter and bridge cases are distinguished according to whether $\delta^{(1)} < \theta_1^{\max}(\mathbf{e}_1)$ holds or not.*

The decay rate of (44) is more informative than its original definition and the other form $\overline{D}(\mathbf{0}, \boldsymbol{\beta})$. However, it remains to identify $\delta^{(i)}$ for $i = 1, 2$. This problem can be reduced to finding the convergence parameters of the operator moment generating functions of the Markov additive kernels, denoted by $J^{(i)}$, of the local processes $\{\boldsymbol{Y}_n^{(i)}\}$ for $i = 1, 2$. Define this operator, i.e., matrix, for $i = 1$ by

$$J_*^{(1)}(\theta)(j, k) = E(e^{\theta Y_{11}^{(1)}} 1(Y_{21}^{(1)} = k) | Y_{10}^{(1)} = 0, Y_{20}^{(1)} = j), \qquad j, k \in \mathbb{Z}_+.$$

Similarly we define $J_*^{(2)}(\theta)$ for $\boldsymbol{Y}_n^{(2)}$.

Lemma 5.3 *Under the moment condition (M1),*

$$\delta^{(i)} = \sup\{\theta \ge 0; c_p(J_*^{(i)}(\theta)) = 1\}, \qquad i = 1, 2. \tag{46}$$

Proof. It is sufficient to prove for $i = 1$. Let

$$H^{(1)}(n) = \sum_{\ell=1}^{\infty} P(\zeta_\ell^{(1)} = n | Y_{10}^{(1)} = Y_{20}^{(1)} = 0), \qquad n \in \mathbb{Z},$$

then

$$\begin{aligned} \sum_{n=-\infty}^{\infty} e^{\theta n} H^{(1)}(n) &= \sum_{n=-\infty}^{\infty} e^{\theta n} \sum_{\ell=1}^{\infty} P(Y_{1\ell}^{(1)} = n, Y_{2\ell}^{(1)} = 0 | Y_{10}^{(1)} = Y_{20}^{(1)} = 0) \\ &= \sum_{\ell=0}^{\infty} [(J^{(i)}(\theta))^\ell]_{00}. \end{aligned}$$

Since $E(\zeta_1^{(1)}) < 0$ or $P(\tau_1^{(1)} < \infty) < 1$, the left-hand side of the above equation converges for some $\theta > 0$, and it converges (diverges) if $c_p(J_*^{(1)}(\theta)) > 1$ (< 1). On the other hand, $\delta^{(1)} = \sup\{t \ge 0; \psi(t) \le 1\}$ is the decay rate of $H^{(1)}(n)$ as $n \to \infty$. Hence, we have (46). □

Finding θ satisfying $c_p(J_*^{(i)}(\theta)) = 1$ is yet another hard problem, but we have already answered it for the double QBD process in Section 3. From Proposition 9 of [18], we can get the same answer when the random walk $\{\boldsymbol{Z}_n\}$ is skip free in the directions toward the boundaries and has bounded jumps in all directions. Thus, we define $\theta_1^{(c)}$ and $\eta_2^{(c)}$ in the same way as in the case of the double QBD process (see Section 4). That is,

Corollary 5.2 *For the* 0*-partially homogenous chain* $\{\boldsymbol{Y}_n\}$ *satisfying the assumptions of Theorem 5.1, if the jumps* $\boldsymbol{X}$ *and* $\boldsymbol{X}^{(i)}$ *for* $i = 1, 2$ *are bounded with probability one and* $P(X_1 < -1$ *or* $X_2 < -1) = 0$*, then*

$$\delta^{(i)} = \begin{cases} \theta_1^{(c)}, & i = 1, \\ \eta_2^{(c)}, & i = 2. \end{cases}$$

Remark 5.7 *Combining this with Corollary 5.1 reproduces the results in [17].*

Remark 5.8 *Let us consider how the optimal path is related to our classifications (C1), (C2) and (C3) for the double QBD process. We first note that Corollaries 5.1 and 5.2 verify* $\lambda_i = \alpha_i$*. Consider the* L_1 *direction. Then, by Remark 5.6, (C1) with* $\theta_1^{(c)} < \theta_1^{\max}$ *(*$\theta_1^{(c)} = \theta_1^{\max}$*) is jitter (bridge), (C2) is cascade, and (C3) is jitter. For the* L_2 *direction, (C2) is jitter, (C3) is cascade and the others are the same. So, the classifications exactly correspond with how the optimal path is chosen for* $D(\mathbf{0}, \mathbf{e}_1)$ *and* $D(\mathbf{0}, \mathbf{e}_2)$*.*

6. Examples. We exemplify the results in the previous sections for the two node Jackson network and its modification when one server at a specified node becomes idle. The Jackson network is well known to have the product form stationary distribution while the modified model does not. The decay rate problem for the latter is considered in [9], but a complete answer has not yet been obtained. Both are toy examples, but we can see not only how the present results work but also answer to some interesting questions. For example, we can see when server cooperation improves the decay rate.

6.1 Two node Jackson network. Consider the two node Jackson queueing network. This network has two nodes numbered as 1 and 2. Node i has exogenous arrivals subject to a Poisson process with rate λ_i, which are independent of everything else. It has a single server whose service times are *i.i.d.* with the exponential distribution with mean $1/\mu_i$. Customers who complete their service at node i go to the other node j with probability r_{ij} or leave the network with probability $1 - r_{ij}$. To formulate this model in discrete time, we assume that $\lambda_1 + \lambda_2 + \mu_1 + \mu_2 = 1$. We also assume that $r_{12}r_{21} < 1$ to exclude trivial cases. Let

$$\rho_i = \frac{\lambda_i + r_{ji}\lambda_j}{(1 - r_{ij}r_{ji})\mu_i}, \qquad i \neq j, i, j = 1, 2.$$

Then, this network has a stationary distribution if and only if $\rho_1 < 1$ and $\rho_2 < 1$, which is assumed below. In this case, the stationary distribution $\{\nu_{mn}\}$ has the product form, namely,

$$\nu_{mn} = (1 - \rho_1)(1 - \rho_2)\rho_1^m \rho_2^n, \qquad m, n \in \mathbb{Z}_+.$$

Thus, we know that the asymptotic decay rates are exactly geometric. What is interesting for us is to see how Theorems 4.1 and 4.2 together with Corollary 4.1 work. In view of Propositions 3.1 and 3.2, this is already answered in Section 5 of [23] and Section 3 of [31] (see Figures 2 and 3 of [23]). We here consider how to apply Theorem 4.2 for the Jackson network. From the proof of Lemma 3.2 of [31], we can see that

$$z = \rho_1^{-1}, \qquad w(z) = \frac{\rho_1}{r_{21} + (1 - r_{21})\rho_1}$$

satisfy (11) and (15) with equality, and the other $w(z)$ is given by ρ_2. Denote the nonzero end point of $\mathcal{D}_1$ by $(\theta_1^{\rm end}, \theta_2^{\rm end})$, then we have

$$\theta_1^{\rm end} = -\log \rho_1, \qquad \theta_2^{\rm end} = -\log \frac{\rho_1}{r_{21} + (1 - r_{21})\rho_1},$$

and, if $\theta_2^{\rm end} < -\log \rho_2$ $(= \eta_2^{\rm end}$ as described later), equivalently,

$$\frac{r_{21}\rho_2}{1 - \rho_2} < \frac{\rho_1}{1 - \rho_1}, \tag{47}$$

then $(\theta_1^{(c)}, \theta_2^{(c)}) = (\theta_1^{\rm end}, \theta_2^{\rm end})$. Otherwise, $(\theta_1^{(c)}, \theta_2^{(c)}) = (\theta_1^{\max}, \theta_2^{\max})$, where $-\log \rho_2 \le \theta_2^{\max} \le \theta_2^{\rm end}$.

Similarly, denoting the nonzero end point of $\mathcal{D}_2$ by $(\eta_1^{\rm end}, \eta_2^{\rm end})$, we have

$$\eta_1^{\rm end} = -\log \frac{\rho_2}{r_{12} + (1 - r_{12})\rho_2}, \qquad \eta_2^{\rm end} = -\log \rho_2,$$

and, if $\eta_1^{\rm end} < -\log \rho_1 (= \theta_1^{\rm end})$, equivalently,

$$\frac{r_{12}\rho_1}{1-\rho_1} < \frac{\rho_2}{1-\rho_2}, \tag{48}$$

then $(\eta_1^{(c)}, \eta_2^{(c)}) = (\eta_1^{\rm end}, \eta_2^{\rm end})$. Otherwise, $(\eta_1^{(c)}, \eta_2^{(c)}) = (\eta_1^{\max}, \eta_2^{\max})$, where $-\log \rho_1 \le \eta_1^{\max} \le \eta_1^{\rm end}$.

Thus, we need to consider the four cases according to whether (47) and (48) are true or not. We first note that at least one is true because if both are false we have

$$r_{12}r_{21} \ge 1,$$

which is impossible by our assumption. Secondly, suppose that both of them are true. Then, $\eta_1^{(c)} = \eta_1^{\rm end} < \theta_1^{\rm end} = \theta_1^{(c)}$ and $\theta_2^{(c)} = \theta_2^{\rm end} < \eta_2^{\rm end} = \eta_2^{(c)}$. Hence, we get case (C1). Since $\theta_1^{\rm end} < \theta_1^{\max}$ and $\eta_2^{\rm end} < \eta_2^{\max}$, the asymptotic decay rates are exactly geometric in both directions by (4c) of Theorem 4.2. Thirdly, suppose that (47) is true but (48) is not true. In this case, we have $\eta_1^{(c)} = \eta_1^{\max} \ge -\log \rho_1 = \theta_1^{\rm end} = \theta_1^{(c)}$ and $\theta_2^{(c)} = \theta_2^{\rm end} < \eta_2^{\rm end} \le \eta_2^{\max} = \eta_2^{(c)}$. Thus, we get (C3), so the asymptotic decay rates are exactly geometric in both directions by Theorem 4.1. Finally, due to the symmetric role of L_1 and L_2 directions, we get (C2) if (47) is not true but (48) is true. Consequently, we have the asymptotically exact geometric decay rates ρ_1 and ρ_2 for the L_1 and L_2 directions, as we expect. Typical figures for (C1) and (C3) are given in Figure 5.

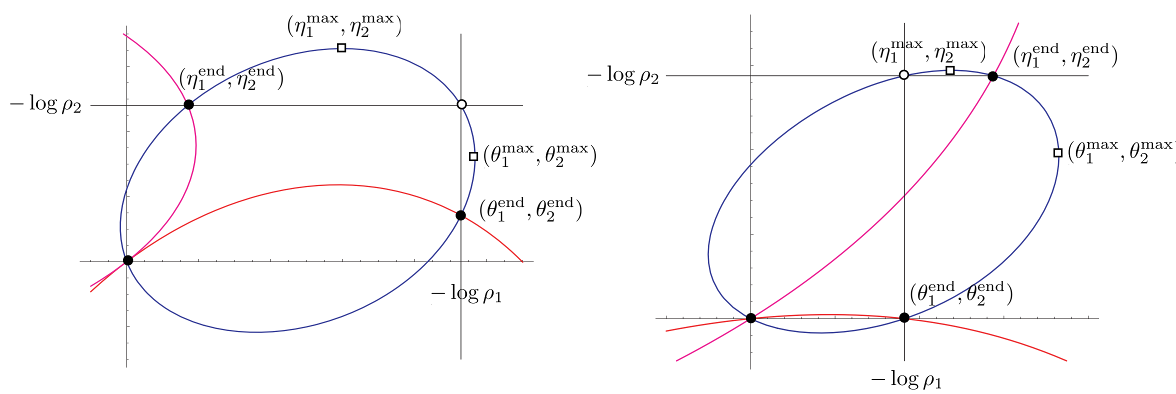

Figure 5: Typical figures for (C1) and (C3)

We finally check Theorem 5.1. In this case, it is easy to see that, for (C1), $\overline{\boldsymbol{\delta}}^{(1)} = \overline{\boldsymbol{\delta}}^{(2)} = (\log \rho_1^{-1}, \log \rho_2^{-1})$, for (C2), $\overline{\boldsymbol{\delta}}^{(1)} = (\log \rho_1^{-1}, \theta_2^{\rm end})$ and $\overline{\boldsymbol{\delta}}^{(2)} = (\log \rho_1^{-1}, \log \rho_2^{-1})$ and, for (C3), $\overline{\boldsymbol{\delta}}^{(1)} = (\log \rho_1^{-1} \log \rho_2^{-1})$ and $\overline{\boldsymbol{\delta}}^{(2)} = (\eta_1^{\rm end}, \log \rho_2^{-1})$. In all the cases, for $\boldsymbol{\beta} = (\beta_1, \beta_2)$,

$$D(\mathbf{0}, \boldsymbol{\beta}) = \beta_1 \log \rho_1^{-1} + \beta_2 \log \rho_2^{-1} = -\log(\rho_1^{\beta_1} \rho_2^{\beta_2}),$$

as expected.

6.2 Modified Jackson network. We next modify the Jackson network in such a way that a server at node 2 helps the service at node 1 when node 2 is empty. We describe this by changing μ_1 in $p_{(-1)0}^{(1)} = \mu_1(1 - r_{12})$ and $p_{(-1)1}^{(1)} = \mu_1 r_{12}$ of the Jackson network into $\mu_1^* \ge \mu_1$. Assume its stability condition that $\rho_2 < 1$ and $\mu_1^* > (\lambda_1 - \mu_1\rho_2)/(1-\rho_2)$ (see [9]). We are interested in when this cooperation of the servers improves the system performance. We consider this by the tail decay rates of the stationary queue length distribution at nodes 1 and 2. Similarly in Section 6.1, we denote the nonzero end points of $\mathcal{D}_1$ and $\mathcal{D}_2$ by $(\theta_1^{\rm end}, \theta_2^{\rm end})$ and $(\eta_1^{\rm end}, \eta_2^{\rm end})$, respectively.

For this modified Jackson network, Foley and McDonald [9] study the decay rates at node 1. They consider the three cases (corresponding sections of [9]):

(FM1) $\rho_2 \le \exp(-\theta_2^{\rm end})$ and $\theta_2^{\rm end} < \theta_2^{\max}$ (Section 7.1),
(FM2) $\rho_2 < \exp(-\theta_2^{\rm end})$ and $\theta_2^{\rm end} > \theta_2^{\max}$ (Section 7.2),
(FM3) $\rho_2 < \exp(-\theta_2^{\rm end})$ and $\theta_2^{\rm end} = \theta_2^{\max}$ (Section 7.3).

Note that $\rho_2 \le \exp(-\theta_2^{\rm end})$ is equivalent to $\theta_2^{\rm end} \le \log(\rho_2^{-1}) = \eta_2^{\rm end}$ since R_2 is unchanged by the modification. Hence, the case where $\theta_2^{\rm end} > \theta_2^{\max}$ is missing here. This is case (C2) in our terminology and the cascade case in large deviations terminology (see Remark 5.8). However, the condition $\rho_2 \le \exp(-\theta_2^{\rm end})$ is crucial in [9].

Thus, if (FM1) and $\theta_1^{\rm end} \le \eta_1^{(c)}$ hold, then we have (C3). If (FM1) and $\theta_1^{\rm end} > \eta_1^{(c)}$ hold, then we have (C1) and (4c). Both cases have asymptotically exact geometric decay rate $e^{-\theta_1^{\rm end}}$, where $\theta_1^{\rm end}$ is θ_1^j of [9]. If (FM2) holds, then we have (C1) and (4d2) of Theorem 4.2. Finally, if (FM3) holds, then we have (C1) and (4d). For both of (FM2) and (FM3), the rough geometric decay rate is $e^{-\theta_1^{\max}}$, where $\theta_1^{\max}$ is θ_1^b of [9]. Hence, Theorem 4.2 implies the exact asymptotics obtained in [9]. Note that Theorem 4.2 gives us more. Namely, if $\rho_2 > \exp(-\theta_2^{\rm end})$, equivalently, $\theta_2^{\rm end} > \eta_2^{\rm end}$, then we have either (C1) or (C2). (C1) occurs if $\theta_2^{(c)} = \theta_2^{\max} < \eta_2^{\rm end} < \theta_2^{\rm end}$, which implies (4d2). On the other hand, (C2) occurs if $\eta_2^{\rm end} \le \min(\theta_2^{\max}, \theta_2^{\rm end})$.

We now consider how the increase of μ_1^* improves the decay rate at node 1. For this, we write (17) with equality in the explicit form

$$\lambda_1 e^{\theta_1} + \mu_1^* r_{12} e^{-\theta_1} + (\lambda_2 + \mu_1^*(1-r_{12})e^{-\theta_1})e^{\theta_2} = \lambda_1 + \lambda_2 + \mu_1^*.$$

Hence,

$$e^{\theta_2} = \frac{\lambda_1(1-e^{\theta_1}) + \lambda_2 + \mu_1^* r_{12}(1-e^{-\theta_1}) + \mu_1^*(1-r_{12})}{\lambda_2 + \mu_1^*(1-r_{12})e^{-\theta_1}}.$$

This means that θ_2 is increased as μ_1^* is increased for each fixed $\theta_1 \ge 0$. Let $\mu_{11}^{\max}$ be the value of μ_1^* for which $\theta_1^{\rm end} = \theta_1^{\max}$, which exists only when (47) holds. Similarly, let $\mu_{12}^{\max}$ be the value of μ_1^* for which $\theta_1^{\rm end} = \eta_1^{\max}$ if it exists. Then, we have the following conclusions.

(J1) Suppose that (47) and (48) hold. Then, increasing μ_1^* decreases the geometric decay rate $e^{-\alpha_1}$ at node 1 as long as $\mu_1 \le \mu_1^* < \mu_{11}^{\max}$, and there is no further improvement on this decay rate for $\mu_1^* \ge \mu_{11}^{\max}$. There is no improvement for the geometric decay rate $e^{-\alpha_2}$ at node 2 for all $\mu_1^* \ge \mu_1$.

(J2) Suppose that (47) holds and (48) does not hold. Then, increasing μ_1^* decreases both decay rate $e^{-\alpha_1}$ and $e^{-\alpha_2}$ as long as $\mu_1 \le \mu_1^* < \mu_{12}^{\max}$, and there is no further improvement on the geometric decay rate $e^{-\alpha_2}$ for $\mu_1^* \ge \mu_{12}^{\max}$ while $e^{-\alpha_1}$ is decreased for $\mu_1^* \in [\mu_{12}^{\max}, \mu_{11}^{\max})$ and remains constant for $\mu_1^* \ge \mu_{11}^{\max}$.

(J3) Suppose that (47) does not hold and (48) holds. Then, there is no further improvement on both geometric decay rates $e^{-\alpha_1}$ and $e^{-\alpha_2}$ for all $\mu_1^* \ge \mu_1$.

Thus, there are the cases that no improvement is made in either geometric decay rate. Condition (47) is a crucial condition for this. This condition may be intuitively interpreted that the congestion at node 1 is greater than the congestion from node 2 to node 1. This intuition supports our conclusions. What is more interesting is that we can mathematically characterize when the decay rates are improved.

7. Extension and concluding remarks. The results of this paper can be applied to find the decay rates as long as a model can be formulated as a double QBD or some extensions such as a two sided double QBD in [25]. In that paper, the author considers the generalized joining the shortest queue with two queues introduced in [7], and derive a complete solution for the rough decay rates.

We can conceive of various extensions of the present results. We first consider removing the irreducibility condition (ii) for the local processes $\{\boldsymbol{L}_n^{(i)}\}$. This condition may be restrictive in applications. For example, a preemptive priority queue does not satisfy it. In what follows, we outline how we can get the rough decay rates without condition (ii). Detailed proofs for this case will be given elsewhere.

Let us consider the case that condition (ii) does not hold for the double QBD process $\{\boldsymbol{L}_n\}$. As usual, we assume the stability condition (i). Recall that $\boldsymbol{X} = (X_1, X_2)$ is a random vector that describes one step movement of $\{\boldsymbol{L}_n\}$ inside the positive quadrant. Then, (ii) does not hold only when at least one of $P(X_1 > 0)$, $P(X_1 < 0)$, $P(X_2 > 0)$, $P(X_2 < 0)$, $P(X_2^{(1)} > 0)$ or $P(X_1^{(2)} > 0)$ vanishes. The last two probabilities only concern $\varphi^{(i)}(\boldsymbol{\theta})$ for $i = 1, 2$, and one of them must be positive because of the irreducibility of $\{\boldsymbol{L}_n\}$.

Let us consider the case that $P(X_2^{(1)} > 0) = 0$. In this case, $\varphi^{(1)}(\boldsymbol{\theta}) = E(e^{\theta_1 X_1^{(1)}})$, and therefore the region of (17) is the stripe between two vertical lines one of which goes through the origin. This stripe is included in the left half space $\theta_1 \le 0$ if $E(X_1^{(1)}) \ge 0$, and in the right half space $\theta_1 \ge 0$ if $E(X_1^{(1)}) < 0$. In either case, we obviously have $(\theta_1, \underline{\theta}_2), (\theta_1, \overline{\theta}_2) \in \mathcal{D}_1$ in the terminology of Theorem 3.1. Hence, the first part of Corollary 3.1 is still valid (see its proof in Appendix C), and we can prove Theorem 4.1 if the first four probabilities are positive since we must have $P(X_1^{(2)} > 0) > 0$.

So, we consider the first four probabilities. In principle, there are sixteen cases in total according to which each of them vanish. However, for each $i = 1, 2$, if both of $P(X_i > 0)$ and $P(X_i < 0)$ vanish, then the double QBD process can be considered as one dimensional process. So, these cases are not considered. Taking the symmetric structure of the double QBD process into account, we only need to consider the following four cases for the decay rate in the L_1-direction.

(K1) $P(X_1 > 0) = 0$ and $P(X_1 < 0), P(X_2 > 0), P(X_2 < 0) > 0$.
(K2) $P(X_1 < 0) = 0$ and $P(X_1 > 0), P(X_2 > 0), P(X_2 < 0) > 0$.
(K3) $P(X_1 > 0) = P(X_2 > 0) = 0$ and $P(X_1 < 0), P(X_2 < 0) > 0$.
(K4) $P(X_1 < 0) = P(X_2 > 0) = 0$ and $P(X_1 > 0), P(X_2 < 0) > 0$.

In all cases, (11) becomes linear concerning $w(z)$, so $A_*^{(1)}(z)$ has only one left invariant positive vector, and we can not use Proposition 3.1 and (3a) of Proposition 3.2. However, (3b) of Proposition 3.2 holds, and Theorem 3.1 is still valid if we replace $\mathcal{V}_{A^{(1)}}$ and $\mathcal{V}_{R_1}$ by the corresponding sets of the subinvariant vectors. Namely, they are

$$\overline{\mathcal{V}}_{A^{(1)}} = \{(z, \boldsymbol{x}); \boldsymbol{x} A_*^{(1)}(z) \le \boldsymbol{x}, z \ge 1, \boldsymbol{x} > \boldsymbol{0}\},$$
$$\overline{\mathcal{V}}_{R_1} = \{(z, \boldsymbol{x}); \boldsymbol{x} z R_1 \le \boldsymbol{x}, z \ge 1, \boldsymbol{x} > \boldsymbol{0}\}.$$

Of course, (3c) and (3f) have to be changed appropriately, and (3e) must be removed while (3d) is valid. Note that $\mathcal{D}_1$ is not a closed curve in this case. Since equation $\varphi(\boldsymbol{\theta}) = 1$ has a single solution $\boldsymbol{\theta}$ for each θ_1, $\mathcal{D}_1$ can be described by $\theta_2 = f(\theta_1)$ for some function f. It is easy to see that f is concave for (K1) and (K3) while it is convex for (K2) and (K4).

Thus, for (K1) and (K3), $c_p(R_1) = \sup\{e^{\theta_1}; \varphi(\boldsymbol{\theta}) = 1\}$. In this case, we denote $\log c_p(R_1)$ by $\theta_1^{(c)}$, and let $\theta_2^{(c)} = -\infty$. For (K2) and (K4), $c_p(R_1) = e^{\theta_1^{(c)}}$, where $\theta_1^{(c)}$ is a positive solution of θ_1 of the equations $\varphi(\boldsymbol{\theta}) = 1$ and $\varphi^{(1)}(\boldsymbol{\theta}) = 1$. Using these notation, we still have Theorem 4.1.

Other than those irreducibility issue, it may be interesting to consider the following problems.

1) It may be questioned whether Theorems 4.1 and 4.2 and Corollaries 4.3 and 4.4 hold for the 0-partially homogenous chain under appropriate moment conditions. For the discrete state case, we need to generalize Proposition 3.2 for non-skip free processes. We conjecture that this is possible. Foley and McDonald [10] are also investigating such a generalization.
2) In applications, it is interesting to consider the case that the double QBD process, or more generally the 0-partially homogenous reflected random walk, is modulated by a finite state Markov chain. For example, this enables us to answer the decay rates in the generalized Jackson network with phase type interarrival and service time distributions, for which the decay rate problem has been only partially answered (see, e.g., [11, 12]). A Markov modulated fluid queueing network can be also considered under this framework.
3) It would also be interesting to extend the results for a two dimensional reflected Levy processes.
4) Of course, the extension to the more than two dimensional case is a challenging problem known to be very hard.

Appendix A. Proof of Lemma 2.3. We use the change of measures for this proof (see, e.g., Section 4 of [26]). For this, we first note the following characterization by subinvariant vectors, which is similarly obtained in the irreducible case, but it is generally not true if nothing is assumed (see Lemma 2.1 and Remark 2.3 of [15]).

Proposition A.1 (Theorem 6.3 of [32]) *For a nonnegative square matrix T which has the same size as A, if T has a single irreducible class, then we have*

$$c_p(T) = \sup\{z \geq 0; z\boldsymbol{x}T \leq \boldsymbol{x}, \boldsymbol{x} \in \mathcal{X}\}.$$

Let $z_A = \sup\{z \geq 1; \boldsymbol{x}A_*^{(1)}(z) \leq \boldsymbol{x}, \boldsymbol{x} > \boldsymbol{0}\}$. We first show that $z_A \leq c_p(R_1)$. Let

$$G_{1*}^-(z) = A_0^{(1)} + R_1 A_{-1}^{(1)} + z^{-1} A_{-1}^{(1)},$$

which is known to be strictly substochastic for $z > 1$ (see [16]). Hence, from (4), we have

$$(I - A_*^{(1)}(z))(I - G_*^-(z))^{-1} = I - zR_1, \qquad z > 1.$$

From this and Proposition A.1, it easily follows that $z_A \leq c_p(R_1)$. We next prove that $c_p(R_1) \leq z_A$. For this, we consider two cases separately. For notational simplicity, let $z = c_p(R_1)$. First assume that zR_1 is recurrent, i.e., $(I - zR_1)^{-1} = \infty$. In this case, there exists a left invariant positive vector $\boldsymbol{x}$ of zR_1. That is,

$$\boldsymbol{x}(I - zR_1) = \boldsymbol{0}$$

Multiplying both sides of (2) by $\boldsymbol{x}$ from the left, we have

$$\boldsymbol{x}R_1 = z^{-1}\boldsymbol{x}\left(z^{-1}A_{-1}^{(1)} + A_0^{(1)} + zA_1^{(1)}\right) = z^{-1}\boldsymbol{x}A_*^{(1)}(z),$$

which yields

$$\boldsymbol{x}(I - A_*^{(1)}(z)) = \boldsymbol{x}(I - zR_1) = \boldsymbol{0}.$$

Hence, by the definition of z_A, we have $z \leq z_A$. We next assume that zR_1 is transient, i.e., $(I - zR_1)^{-1} < \infty$. In this case, from (4), we have

$$(I - zR_1)^{-1}(I - A_*^{(1)}(z)) = I - G_*^-(z)$$

Note that all off-diagonal entries of the left-hand side matrix of this equation are non positive and the diagonal entries are all positive due to the right-hand side matrix. Since $G_*^-(z)$ is substochastic, its convergence parameter is not less than 1. Hence, from Proposition A.1, we can find a positive row vector $\boldsymbol{y}$ such that

$$\boldsymbol{y}(I - G_*^-(z)) \geq \boldsymbol{0}.$$

Thus, we have

$$\boldsymbol{y}\left((I - zR_1)^{-1}(I - A_*^{(1)}(z))\right) = \boldsymbol{y}(I - G_*^-(z)) \geq \boldsymbol{0}. \tag{49}$$

As we noted, the matrix $(I - zR_1)^{-1}(I - A_*^{(1)}(z))$ has the diagonal entries all of which are positive and the off-diagonal entries all of which are negative. From this fact, we can see that

$$\boldsymbol{y}(I - zR_1)^{-1}A_*^{(1)}(z) < \infty.$$

Hence, $\boldsymbol{y}(I - zR_1)^{-1}$ must be a finite positive vector, so (49) implies that $z \leq z_A$. Thus, we have $c_p(R_1) = z \leq z_A$ for both cases. This completes the proof of Lemma 2.3.

Appendix B. Proof of Proposition 3.2. From the matrix geometric form (3) and the fact that $\boldsymbol{x}$ is also the left eigenvector of zR_1, we have, for any $n \geq 1$,

$$\begin{aligned} z^n \boldsymbol{\nu}_n \boldsymbol{u} &= z \sum_{k\geq 0}\sum_{\ell\geq 0} \frac{\nu_{1k}}{x_k} x_k z^{n-1} [R_1^{n-1}]_{k\ell} u_\ell \\ &= z \sum_{\ell\geq 0}\sum_{k\geq 0} \frac{\nu_{1k}}{x_k} x_\ell u_\ell \frac{1}{x_\ell} z^{n-1} [(R_1^{\mathrm{T}})^{n-1}]_{\ell k} x_k, \end{aligned}$$

where R_1^{T} is the transpose of R_1. Then, denoting the matrix $\{x_\ell^{-1} z [R_1^{\mathrm{T}}]_{\ell k} x_k; \ell, k \in \mathbb{Z}_+\}$ by Q, we get

$$z^n \boldsymbol{\nu}_n \boldsymbol{u} = z \sum_{\ell\geq 0}\sum_{k\geq 0} \frac{\nu_{1k}}{x_k} x_\ell u_\ell [Q^{n-1}]_{\ell k}. \tag{50}$$

From assumption (ii), R_1 and therefore Q are irreducible. Hence, there can be two cases where Q, equivalently, R_1 is positive recurrent.

First consider case (3a). In this case, $\boldsymbol{x}\boldsymbol{y} < \infty$ implies that $\boldsymbol{x}\boldsymbol{q} < \infty$ and Q is positive recurrent by Lemma 4.1 of [26]. Denote the stationary distribution of Q by $\boldsymbol{\pi} = \{\pi_k\}$, where $\pi_k = \dfrac{x_k r_k}{\boldsymbol{x}\boldsymbol{q}}$ for the right invariant vector $\boldsymbol{q} = \{r_k\}$ of zR_1. Since $\overline{d}(z,\boldsymbol{x}) < \infty$ implies that

$$\sup \frac{\nu_{1k}}{x_k} < \infty,$$

it follows from $\boldsymbol{x}\boldsymbol{u} < \infty$ and the bounded convergence theorem that

$$\lim_{n\to\infty} z^n \boldsymbol{\nu}_n \boldsymbol{u} = z\boldsymbol{x}\boldsymbol{u} \sum_{k\ge 0} \frac{\nu_{1k}}{x_k}\pi_k = z\boldsymbol{x}\boldsymbol{u}\frac{\boldsymbol{\nu}_1 \boldsymbol{q}}{\boldsymbol{x}\boldsymbol{q}}.$$

Thus, we get (9).

We next consider case (3b). If Q is positive recurrent, we have already proved the stronger result (9). So, we can assume that Q is not positive recurrent. From the assumption $\overline{d}(z,\boldsymbol{x}) < \infty$, there exists an integer k_0 for any $\epsilon > 0$ such that

$$\frac{\nu_{1k}}{x_k} \le \overline{d}(z,\boldsymbol{x}) + \epsilon, \qquad k > k_0.$$

Hence, we have, from (50), for any $n \ge 1$,

$$\begin{aligned} z^n \boldsymbol{\nu}_n \boldsymbol{u} &\le z \sum_{\ell\ge 0} \left(\sup_{j\le k_0} \frac{\nu_{1j}}{x_j} \sum_{k\le k_0} x_\ell u_\ell [Q^{n-1}]_{\ell k} + (\overline{d}(z,\boldsymbol{x}) + \epsilon) \sum_{k\ge 0} x_\ell u_\ell [Q^{n-1}]_{\ell k} \right) \\ &= z \sum_{\ell\ge 0} \left(\sup_{j\le k_0} \frac{\nu_{1j}}{x_j} \sum_{k\le k_0} x_\ell u_\ell [Q^{n-1}]_{\ell k} \right) + z(\overline{d}(z,\boldsymbol{x}) + \epsilon) \sum_{\ell\ge 0} x_\ell u_\ell, \end{aligned}$$

where the last equality is obtained since Q is stochastic. Since Q is not positive recurrent, this and the dominated convergence theorem with $\boldsymbol{x}\boldsymbol{u} < \infty$ yield

$$\limsup_{n\to\infty} z^n \boldsymbol{\nu}_n \boldsymbol{u} \le z\overline{d}(z,\boldsymbol{x})\boldsymbol{x}\boldsymbol{u}.$$

Thus, letting $\overline{d}^{\dagger}(\boldsymbol{x}) = \max\left(\sum_{k\ge 0} \nu_{1k} x_k^{-1} \pi_k, \overline{d}(z,\boldsymbol{x})\right)$, we have the last inequality of (10).

Appendix C. Proof of Corollary 3.1. By Proposition 3.2, the first part is obtained if we show that, for a $(\theta_1,\theta_2) \in \mathcal{D}_1$,

$$\limsup_{n\to\infty} \nu_{1n} e^{\theta_2 n} < \infty,$$

implies that there is $\boldsymbol{x} \in \mathcal{X}$ for $z = e^{\theta_1}$ such that $(z,\boldsymbol{x}) \in \mathcal{V}_{A^{(1)}}$ and

$$\limsup_{n\to\infty} \nu_{1n} x_n^{-1} < \infty.$$

This is immediate from Theorem 3.1 if $(\theta_1, \underline{\theta}_2) \in \mathcal{D}_1$. So, we prove this fact. Let $\varphi_i^{(1)}(\theta_1) = E(e^{\theta_1 X_1^{(1)}}; X_2^{(1)} = i)$, then, for $\boldsymbol{\theta} = (\theta_1,\theta_2)$,

$$\varphi^{(1)}(\boldsymbol{\theta}) = \varphi_0^{(1)}(\theta_1) + e^{\theta_2}\varphi_1^{(1)}(\theta_1).$$

Since $P(X_2^{(1)} = 1) > 0$ by condition (ii), the boundary of (17), that is, $\{\boldsymbol{\theta} \in \mathbb{R}^2; \varphi^{(1)}(\boldsymbol{\theta}) = 1\}$, is not a vertical line, and can be described by $\theta_2 = f(\theta_1)$ using a function f. Obviously, f is either convex or concave. If $\mathcal{D}_1 = \{(0,0)\}$, then the proof is obvious. So, we assume that $\mathcal{D}_1 \ne \{(0,0)\}$. We first show that f is concave. From $\varphi^{(1)}(\boldsymbol{\theta}) = 1$, we have

$$\frac{\partial}{\partial\theta_1}\varphi_0^{(1)}(\theta_1) + e^{\theta_2}\frac{\partial}{\partial\theta_1}\varphi_1^{(1)}(\theta_1) + e^{\theta_2}\varphi_1^{(1)}(\theta_1)\frac{\partial\theta_2}{\partial\theta_1} = 0, \tag{51}$$

$$\begin{aligned} &\frac{\partial^2}{\partial^2\theta_1}\varphi_0^{(1)}(\theta_1) + e^{\theta_2}\frac{\partial^2}{\partial^2\theta_1}\varphi_1^{(1)}(\theta_1) + 2e^{\theta_2}\frac{\partial}{\partial\theta_1}\varphi_1^{(1)}(\theta_1)\frac{\partial\theta_2}{\partial\theta_1} \\ &\qquad + e^{\theta_2}\varphi_1^{(1)}(\theta_1)\frac{\partial^2\theta_2}{\partial^2\theta_1} + e^{\theta_2}\varphi_1^{(1)}(\theta_1)\left(\frac{\partial\theta_2}{\partial\theta_1}\right)^2 = 0. \end{aligned} \tag{52}$$

Since

$$\frac{\partial}{\partial \theta_1}\varphi_i^{(1)}(\theta_1)\Big|_{(\theta_1,\theta_2)=(0,0)} = E(X_1^{(1)}; X_2^{(1)} = i), \qquad i = 0,1,$$

$$\frac{\partial^2}{\partial^2 \theta_1}\varphi_i^{(1)}(\theta_1)\Big|_{(\theta_1,\theta_2)=(0,0)} = E((X_1^{(1)})^2; X_2^{(1)} = i), \qquad i = 0,1,$$

(51) and (52) with $\theta_1 = 0$ and $\theta_2 = 0$ are written as

$$E(X_1^{(1)}) + P(X_2^{(1)} = 1)f'(0) = 0,$$
$$E((X_1^{(1)})^2) + 2E(X_1^{(1)}; X_2^{(1)} = 1)f'(0) + P(X_2^{(1)} = 1)(f''(0) + (f'(0))^2) = 0.$$

Since $P(X_2^{(1)} = 1) > 0$, we get

$$\begin{aligned} f'(0) &= -\frac{E(X_1^{(1)})}{P(X_2^{(1)} = 1)}, \\ f''(0) &= -\frac{1}{P(X_2^{(1)} = 1)}\Big(E((X_1^{(1)})^2) - 2E(X_1^{(1)}; X_2^{(1)} = 1)E(X_1^{(1)}) + E^2(X_1^{(1)})\Big) \\ &= -\frac{1}{P(X_2^{(1)} = 1)}\Big(E((X_1^{(1)} - E(X_1^{(1)}))^2) + 2E(X_1^{(1)}; X_2^{(1)} = 0)E(X_1^{(1)})\Big) < 0. \end{aligned}$$

Hence, f is concave. Note that $f(0) = 0$ and $\{\boldsymbol{\theta}; \varphi(\boldsymbol{\theta}) \le 1\}$ is convex in the θ_1-θ_2 plane and contains $(0,0)$ on its boundary. Since $\mathcal{D}_1$ is not singleton, the lower part of the convex curve (16) from the origin to $(c_p(R_1), f(c_p(R_1))$ is included in $\mathcal{D}_1$ due to the concavity of f. Hence, $(\theta_1,\theta_2) \in \mathcal{D}_1$ implies $(\theta_1, \underline{\theta}_2) \in \mathcal{D}_1$. Thus, we get the desired result. The remaining part is immediate from Lemma 2.1 and Proposition 3.2.

Appendix D. Proof of Corollary 4.2. We first consider the case that $m_1 < 0$ and $m_2 < 0$. If $m_2^{(1)} = 0$, then we have $m_1^{(1)} < 0$ if and only if the first inequality of (S1) holds. As we discussed in Appendix C for the proof of Corollary 3.1, the region of (17) is the closed stripe surrounded by two vertical lines which go through the origin and some other point $(\theta_1, 0)$. This θ_1 is positive if and only if $m_1^{(1)} < 0$. Since the $\theta_1 > 0$ implies that $\mathcal{D}_1$ has a point (θ_1,θ_2) with $\theta_1 > 0$, the first inequality of (S1) holds if and only if $\alpha_1 > 0$.

If $m_2^{(1)} \ne 0$, then $m_2^{(1)} = P(X_2^{(1)} = 1) > 0$. Hence, for (θ_1,θ_2) satisfying (16), we have

$$\frac{\partial \theta_2}{\partial \theta_1}\Big|_{(\theta_1,\theta_2)=(0,0)} = -\frac{m_1}{m_2}.$$

Similarly, for (θ_1', θ_2') satisfying the boundary of (17),

$$\frac{\partial \theta_2'}{\partial \theta_1'}\Big|_{(\theta_1',\theta_2')=(0,0)} = -\frac{m_1^{(1)}}{m_2^{(1)}}.$$

Since $m_2 < 0$ and $m_2^{(1)} > 0$, the first inequality of (S1) is equivalent to $-\frac{m_1}{m_2} < -\frac{m_1^{(1)}}{m_2^{(1)}}$, and therefore equivalent to

$$\frac{\partial \theta_2}{\partial \theta_1}\Big|_{(\theta_1,\theta_2)=(0,0)} < \frac{\partial \theta_2'}{\partial \theta_1'}\Big|_{(\theta_1',\theta_2')=(0,0)}. \tag{53}$$

This implies that $\mathcal{D}_1$ again has a point (θ_1,θ_2) with $\theta_1 > 0$, so $\alpha_1 > 0$. Hence, the first inequality of (S1) holds if and only if $\alpha_1 > 0$. Similarly, we can prove that the second inequality of (S1) holds if and only if $\alpha_2 > 0$.

We next consider case that $m_1 \ge 0$ and $m_2 < 0$. In this case, $-\frac{m_1}{m_2} \ge 0$, and $m_1^{(1)} < 0$ for $m_2^{(1)} = 0$. Note that the second derivative of the curve described by $\varphi(\boldsymbol{\theta}) = 1$ is positive (see Figure 6). Hence, similarly to the above arguments, we can see that the inequality of (S2) holds if and only if $\alpha_1 > 0$. In this case, $\mathcal{D}_2$ always has a point (θ_1,θ_2) with $\theta_2 > 0$ since $\varphi^{(2)}(\boldsymbol{\theta}) = 1$ is a convex curve with nonnegative derivative at the origin. So, we always have $\alpha_2 > 0$.

There remains the case that $m_1 < 0$ and $m_2 \ge 0$, but this case is symmetric with the previous case. Hence, the inequality of (S3) holds if and only if $\alpha_2 > 0$ and we always have $\alpha_1 > 0$. This completes the proof of Corollary 4.2.

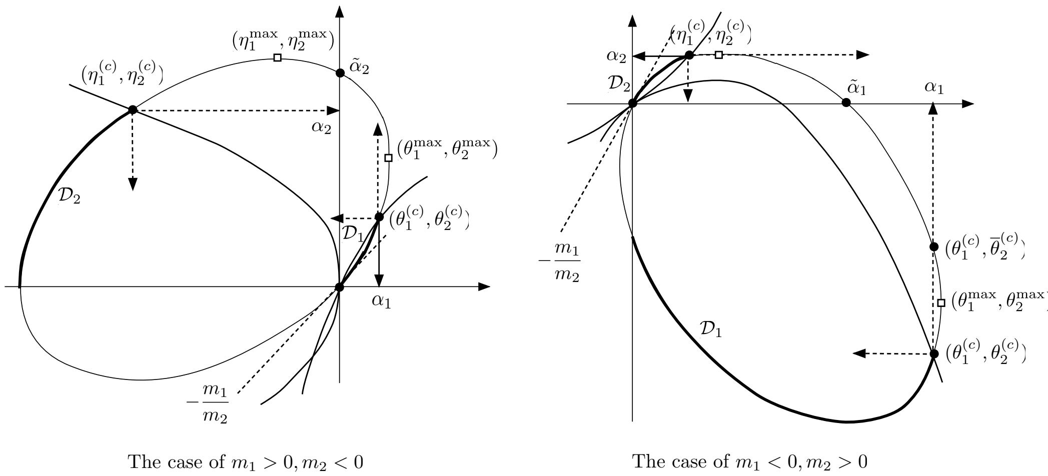

Figure 6: Typical figures for the case that $m_1/m_2 < 0$

Appendix E. Proof of Lemma 5.2. We only prove (40) for $i = 1$ since (40) is symmetric for $i = 1, 2$. Fix $\boldsymbol{\beta} \equiv (\beta_1, \beta_2) \in \mathbb{Z}_+^2$, and define $F(v, \boldsymbol{\theta})$ as

$$F(v, \boldsymbol{\theta}) = uv + (\beta_1 - v)\theta_1 + \beta_2\theta_2, \qquad v \geq 0, \boldsymbol{\theta} \equiv (\theta_1, \theta_2) \in \mathcal{D}_0.$$

Then, by the definitions of $h^{(1)}(u, \boldsymbol{\beta})$ and $D(\boldsymbol{\theta})$ and condition (M2), we have

$$h^{(1)}(u, \boldsymbol{\beta}) = \min_{v \geq 0} \max_{(\theta_1, \theta_2) \in \mathcal{D}_0} F(v, \boldsymbol{\theta}).$$

For each fixed v, let us consider to maximize $F(v, \boldsymbol{\theta})$ under constrain $\boldsymbol{\theta} \in \mathcal{D}_0$. Note that there can be two θ_2 for each θ_1 since $\boldsymbol{\theta} \in \mathcal{D}_0$. However, we have to choose $\overline{\theta}_2$ defined for each θ_1 as

$$\overline{\theta}_2 = \max\{\theta_2; (\theta_1, \theta_2) \in \mathcal{D}_0\},$$

for maximizing $F(v, \boldsymbol{\theta})$. In what follows, we always use this version of $F(v, \boldsymbol{\theta})$.

We separately consider two cases whether $\beta_2 = 0$ or not. First assume that $\beta_2 = 0$. In this case, $F(v, \boldsymbol{\theta}) = uv + (\beta_1 - v)\theta_1$, which implies that

$$\max_{(\theta_1, \theta_2) \in \mathcal{D}_0} F(v, \boldsymbol{\theta}) = \begin{cases} uv, & v \geq \beta_1, \\ uv + (\beta_1 - v)\theta_1^{\max}, & v < \beta_1. \end{cases}$$

Hence, $h^{(1)}(u, \boldsymbol{\beta}) = u\beta_1 = \boldsymbol{\beta}\overline{\boldsymbol{u}}^{(1)}$. Since $u \leq \theta_1^{\max} = \theta_1^{\max}(\boldsymbol{\beta})$, we have (40).

We next consider the case that $\beta_2 \neq 0$. Applying the Karush-Kuhn-Tucker necessary condition to the Lagrangian:

$$L(\boldsymbol{\theta}, \lambda) = uv + (\beta_1 - v)\theta_1 + \beta_2\theta_2 + \lambda(1 - \varphi(\boldsymbol{\theta})),$$

we have

$$\beta_1 - v - \lambda \frac{\partial \varphi}{\partial \theta_1}(\boldsymbol{\theta}) = 0, \qquad \beta_2 - \lambda \frac{\partial \varphi}{\partial \theta_2}(\boldsymbol{\theta}) = 0.$$

Since $\beta_2 \neq 0$ implies $\lambda \neq 0$, we have

$$\beta_2 \frac{\partial \varphi}{\partial \theta_1}(\boldsymbol{\theta}) + (v - \beta_1)\frac{\partial \varphi}{\partial \theta_2}(\boldsymbol{\theta}) = 0. \tag{54}$$

Let $\boldsymbol{\theta}(v) \equiv (\theta_1(v), \theta_2(v))$ be the solution of this maximization. From (54), at point $\boldsymbol{\theta}(v)$, the vector which is normal to the curve $\mathcal{D}_0$ must be proportional to vector $(\beta_1 - v, \beta_2)$. Thus, when v increases from 0 to infinity, the $\boldsymbol{\theta}(v)$ moves on the curve $\mathcal{D}_0$ counterclockwise from $\boldsymbol{\theta}^{\max}(\boldsymbol{\beta})$ to $\boldsymbol{\theta}^{\min}(\mathbf{e}_1) \equiv \arg \min_{\boldsymbol{\theta} \in \mathcal{D}_0} \theta_1$. It is also not hard to see that each entry of $\boldsymbol{\theta}(v)$ is continuously differentiable in v. Hence, $\boldsymbol{\theta}(v) \in \mathcal{D}_0$ yields

$$\theta_1'(v)\frac{\partial \varphi}{\partial \theta_1}(\boldsymbol{\theta}(v)) + \theta_2'(v)\frac{\partial \varphi}{\partial \theta_2}(\boldsymbol{\theta}(v)) = 0. \tag{55}$$

For minimizing $F(v, \boldsymbol{\theta}(v))$, we take its derivative.

$$\frac{d}{dv} F(v, \boldsymbol{\theta}(v)) = u - \theta_1(v) + (\beta_1 - v)\theta_1'(v) + \beta_2 \theta_2'(v).$$

Let us compute $(\beta_1 - v)\theta_1'(v) + \beta_2\theta_2'(v)$. From (54) and (55),

$$((\beta_1 - v)\theta_1'(v) + \beta_2\theta_2'(v)) \frac{\partial \varphi}{\partial \theta_1}(\boldsymbol{\theta}(v)) = \theta_2'(v) \left(\beta_2 \frac{\partial \varphi}{\partial \theta_1}(\boldsymbol{\theta}(v)) + (v - \beta_1) \frac{\partial \varphi}{\partial \theta_2}(\boldsymbol{\theta}(v)) \right) = 0,$$

$$((\beta_1 - v)\theta_1'(v) + \beta_2\theta_2'(v)) \frac{\partial \varphi}{\partial \theta_2}(\boldsymbol{\theta}(v)) = -\theta_1'(v) \left(\beta_2 \frac{\partial \varphi}{\partial \theta_1}(\boldsymbol{\theta}(v)) + (v - \beta_1) \frac{\partial \varphi}{\partial \theta_2}(\boldsymbol{\theta}(v)) \right) = 0.$$

Since $\boldsymbol{\theta}(v) \in \mathcal{D}_0$, $\frac{\partial \varphi}{\partial \theta_1}(\boldsymbol{\theta}(v)) = \frac{\partial \varphi}{\partial \theta_2}(\boldsymbol{\theta}(v)) = 0$ is impossible, so we have $(\beta_1 - v)\theta_1'(v) + \beta_2\theta_2'(v) = 0$. This concludes

$$\frac{d}{dv} F(v, \boldsymbol{\theta}(v)) = u - \theta_1(v).$$

Note that $F(v, \boldsymbol{\theta}(v))$ is convex in v by its definition, $\max_{(\theta_1,\theta_2) \in \mathcal{D}_0} F(v, \boldsymbol{\theta})$, and

$$F(0, \boldsymbol{\theta}(0)) = \max_{\boldsymbol{\theta} \in \mathcal{D}_0} F(0, \boldsymbol{\theta}) = \max_{\boldsymbol{\theta} \in \mathcal{D}_0} \boldsymbol{\beta}\boldsymbol{\theta}^{\mathrm{T}} = D(\boldsymbol{\beta}) > 0.$$

Hence, if there is a $v \geq 0$ such that $\theta_1(v) = u$, then $F(v, \boldsymbol{\theta}(v))$ is minimized at this v. From the observation that we made for $\boldsymbol{\theta}(v)$ movement, this holds only if $u \leq \theta_1^{\max}(\boldsymbol{\beta})$. In this case, we have

$$h^{(1)}(u, \boldsymbol{\beta}) = F(v, \overline{\boldsymbol{u}}^{(1)}) = \boldsymbol{\beta}\overline{\boldsymbol{u}}^{(1)},$$

since $\theta_1(v) = u$ implies $\theta_2(v) = \overline{u}_2^{(1)}$. Otherwise, $F(v, \boldsymbol{\theta}(v))$ is minimized at $v = 0$ since it can be arbitrarily large for large v from the fact that $\max_{\boldsymbol{\theta} \in \mathcal{D}_0} F(v, \boldsymbol{\theta}) \geq F(v, \mathbf{0}) = uv$. Hence,

$$h^{(1)}(u, \boldsymbol{\beta}) = F(0, \boldsymbol{\theta}(0)) = D(\boldsymbol{\beta}).$$

Thus, we have (40), which completes the proof.

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