Asymptotic behaviors of the loss rate for Markov modulated fluid queue with a finite buffer

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Abstract

We consider a Markov modulated fluid queue with a finite buffer. It is assumed that the fluid flow is modulated by a background Markov chain which may have different transitions when the buffer content is empty or full. In [13], we have studied asymptotic loss rate for this type of fluid queue when the mean drift of the fluid flow is negative. However, the null drift case is not studied. Our major interest is in asymptotic loss rate of the fluid queue with a finite buffer including the null drift case. We consider the density of the stationary buffer content distribution, and derive it in certain closed form from an occupation measure. This result is not only useful to get the asymptotic loss rate especially for the null drift case, but also it is interesting in its own light.

Keywords: Fluid queue, finite buffer, Markov modulated, loss rate, asymptotic behavior, null drift.

1 Introduction

We consider a stochastic fluid queue with a finite buffer, where the fluid flow is modulated by a background Markov chain with a finite state space. It is assumed that the transition probability of the background Markov chain changes when the buffer content is empty or full (e.g., see [4] and [7]). In general, it is difficult to obtain stationary characteristics such as loss rate in analytically tractable form for a fluid queue with a finite buffer. So, there have been some researches to study their asymptotic behavior as the buffer size gets large (e.g., see [3], [13] and references therein). The objective of this paper is two fold. We aim to derive asymptotic loss rate of the Markov modulated fluid queue with a finite buffer. When the mean drift of the net fluid flow is not null, this is studied in [13] (also see [3] for a related model). However, the null drift case is never considered to the best of our knowledge. In this paper, we include this null drift case. For this, we consider the density of the stationary buffer content distribution, and derive it in certain closed form. It turns out that this density is particularly useful to get the asymptotic loss rate for the null drift case. Thus, deriving the stationary density in tractable form is our second purpose.

The stationary density of the fluid queue has been studied by da Silva Soares and Latouche [4] when the mean drift is not null. They first intuitively derive the density assuming its existence, and using some results in [12], then verify that it satisfies differential equations for the stationary distribution. In this way, they get the stationary density in terms of combination of matrix exponentials for the non-null drift case. However, this approach can not be used for the null drift case (see Theorem 4.4 in [5]).

In this paper, we take a different way so as to include the null drift case. We use occupation measures of the Markov additive process which generates the buffer content process at off boundaries. In our formulation, the existence of the density is verified through a occupation measure, which is valid for the null drift case. Furthermore, the stationary density is obtained in terms of hitting probabilities of a certain time-reversed process. This is useful to derive the asymptotic loss rate especially for the null drift case.

Fluid queues have been studied in much literature (e.g., see [2], [5], [6], [12] and references therein). A famous example among them is a fluid queue with on-off sources, which has been studied in [2] and many other papers. This model is a special case of a Markov modulated fluid queue with an infinite buffer. For this model, the stationary distribution of the buffer content has been obtained in several ways, e.g, using the spectral expansions, martingale, the matrix analytic method and the Wiener-Hopf factorization. Recently, Asmussen and Pihlsgård [3] studied the asymptotic behavior of a Levy Process with two reflecting boundaries, and generalized it to a Markov modulated Levy Process. They derived the asymptotic loss rate when the mean drift is not null. This model assumes that the background process is independent of the buffer content, so it has the simpler transition structure at the boundaries while its net fluid flow is more complicated.

As is well known, there is similarity between quasi-birth-and-death process (QBD, for short) process and the Markov modulated fluid queues (e.g., see [12]). In this sense, our work is related to asymptotic study for a QBD process with a finite buffer (e.g., see [11]). In particular, our arguments for the null drift case is inspired by the results in Kim, Kim and Lee [8], in which a finite buffer QBD process is considered when the traffic intensity is unit.

The rest of this paper is organized in the following way. In Section 2, we introduce the Markov additive process, and discuss its basic properties which will be used in the subsequent sections. In Section 3 and 4, using the Markov additive process, we describe the Markov modulated fluid queues with a finite and an infinite buffers, respectively. We finally derive the tail asymptotics of the loss rate for the Markov modulated fluid queue with a finite buffer in Section 5.

2 Markov additive process for fluid queues

We introduce a Markov additive process, which will be used to generate the Markov modulated fluid queues in the subsequent sections. We are interested in two basic quantities, occupation measure and hitting probability under taboo sets, and derive key relations among them. In the subsequent sections, they are used to get the tail asymptotics of the fluid queue with a finite buffer.

Throughout this paper, we shall use the following notations. For matrix A and vector $\boldsymbol{a}$, denote their (i,j)-th and i-th elements by $[A]_{ij}$ and $[\boldsymbol{a}]_i$, respectively. Let $\Delta_{\boldsymbol{a}}$ be the diagonal matrix whose (i,i)-th entry is the i-th entry of vector $\boldsymbol{a}$. Denote the transpositions for matrix A and vector $\boldsymbol{a}$ by A^{T} and $\boldsymbol{a}^{\mathrm{T}}$, respectively. Let $\mathbf{1}$ be the unit vector whose elements all equal one, where its size can be identified in the context where it appears. Denote the number of elements in set $\mathcal{A}$ by $|\mathcal{A}|$. For vector $\boldsymbol{a} \equiv (a_i; i \in \mathcal{A})$, denote the vector whose i-th component is $|a_i|$ by $|\boldsymbol{a}|$, and denote its norm by $\|\boldsymbol{a}\| = \sum_{i \in \mathcal{A}} |a_i|$.

We define the Markov additive process in the following way. Let $M(t)$ be a continuous time Markov chain with finite state space $\mathcal{S}$. Denote the transition rate matrix of $M(t)$ by the $|\mathcal{S}| \times |\mathcal{S}|$ matrix C. Assume that $M(t)$ is irreducible, then there exists the stationary distribution $\boldsymbol{\pi}$ satisfying $\boldsymbol{\pi} C = \mathbf{0}$ and $\boldsymbol{\pi}\mathbf{1} = 1$. Let $X(t)$ be a cumulative process whose rate at time t is given by

$$\frac{d}{dt} X(t) = r(M(t)),$$

where r is a real-valued function defined on $\mathcal{S}$, and r is referred to as a net flow rate. Then $(X(t), M(t))$ is a two-dimensional continuous time Markov chain with state space $\mathbb{R} \times \mathcal{S}$, where $\mathbb{R}$ is the set of all real numbers. We refer to $(X(t), M(t))$ as a Markov additive process, where $X(t)$ and $M(t)$ are called level and background processes, respectively.

Let $\boldsymbol{r}$ be the $|\mathcal{S}|$-dimensional column vector whose i-th element is $r(i)$. For simplicity, we assume that r takes non-zero value. We divide the state space of the background process into two disjoint subsets as follows :

$$\mathcal{S}^- = \{i \in \mathcal{S} | r(i) < 0\}, \quad \mathcal{S}^+ = \{i \in \mathcal{S} | r(i) > 0\}.$$

To avoid trivial case, we assume that neither $\mathcal{S}^-$ nor $\mathcal{S}^+$ is empty. We partition C, $\boldsymbol{\pi}$ and $\boldsymbol{r}$ according to $\mathcal{S}^-$ and $\mathcal{S}^+$ in the following way.

$$C = \begin{pmatrix} C^{--} & C^{-+} \\ C^{+-} & C^{++} \end{pmatrix}, \quad \boldsymbol{\pi} = (\boldsymbol{\pi}^-, \boldsymbol{\pi}^+), \quad \boldsymbol{r} = \begin{pmatrix} \boldsymbol{r}^- \\ \boldsymbol{r}^+ \end{pmatrix}.$$

Then, the mean drift of the Markov additive process is given by

$$d = \boldsymbol{\pi}\boldsymbol{r} (= \boldsymbol{\pi}^+\boldsymbol{r}^+ + \boldsymbol{\pi}^-\boldsymbol{r}^-).$$

We now introduce occupation measure and hitting probability under taboo levels for this Markov additive process. Define the hitting time to level $a \in \mathbb{R}$ as

$$\tau_a = \inf\{t > 0 | X(t) = a\},$$

and denote the hitting time to level a with background state $i \in \mathcal{S}$ as

$$\tau_a(i) = \inf\{t > 0 | X(t) = a, M(t) = i\}.$$

For $a, b \in \mathbb{R}$, let $\tau_{ab} = \tau_a \wedge \tau_b$.

Definition 2.1 For $a, x, y \in \mathbb{R}$, let $\Phi_a(x,y)$ be the $|\mathcal{S}| \times |\mathcal{S}|$ matrix whose (i,j)-th element is given by

$$[\Phi_a(x,y)]_{ij} = E\left[\int_0^{\tau_a} 1(X(t) \in (0,y], M(t) = j) dt \middle| X(0) = x, M(0) = i \right],$$

which is referred to as the occupation measure of subspace $(0, y] \times \{j\}$ before hitting level a, starting from level x with background state i. When τ_a is replaced by τ_{ab} for $a, b \in I\!R$, we denote the corresponding occupation measure by $\Phi_{ab}(x, y)$. Let $\Psi_a(x, y)$ be the $|\mathcal{S}| \times |\mathcal{S}|$ matrix whose (i, j)-th element is given by

$$[\Psi_a(x, y)]_{ij} = P(M(\tau_y) = j, \tau_y \le \tau_a, \tau_y < \infty | X(0) = x, M(0) = i),$$

which is referred to as the hitting probability for level y with background state j before hitting level a, starting from level x with background state i.

We partition $\Phi_a(x, y)$, $\Phi_{ab}(x, y)$ and $\Psi_a(x, y)$ according to $\mathcal{S}^-$ and $\mathcal{S}^+$ as follows.

$$\Phi_a(x, y) = \begin{pmatrix} \Phi_a^{--}(x, y) & \Phi_a^{-+}(x, y) \\ \Phi_a^{+-}(x, y) & \Phi_a^{++}(x, y) \end{pmatrix}, \quad \Phi_{ab}(x, y) = \begin{pmatrix} \Phi_{ab}^{--}(x, y) & \Phi_{ab}^{-+}(x, y) \\ \Phi_{ab}^{+-}(x, y) & \Phi_{ab}^{++}(x, y) \end{pmatrix},$$
$$\Psi_a(x, y) = \begin{pmatrix} \Psi_a^{--}(x, y) & \Psi_a^{-+}(x, y) \\ \Psi_a^{+-}(x, y) & \Psi_a^{++}(x, y) \end{pmatrix}.$$

Furthermore, for $a \in I\!R$, we denote

$$\Psi^{-+} = \Psi_a^{-+}(a, a), \quad \Psi^{+-} = \Psi_a^{+-}(a, a),$$

since they are independent of a.

In what follows, we show that the occupation measure has a density. To this end, we first note that for $x > 0$, the hitting probability $\Psi_{-\infty}^{\bullet +}(0, x)$ has the following matrix exponential form by Theorem 3.1 of [10].

Proposition 2.1 For $x > 0$, we have

$$\Psi_{-\infty}^{\bullet +}(0, x) = \begin{pmatrix} \Psi^{-+} \exp(xQ^{(+)}) \\ \exp(xQ^{(+)}) \end{pmatrix}, \tag{2.1}$$

where Ψ^{-+} and $Q^{(+)}$ are the minimal solutions of the following equations.

$$Q^{(+)} = \Delta_{\boldsymbol{r}^+}^{-1}(C^{++} + C^{+-}\Psi^{-+}), \quad \Psi^{-+}Q^{(+)} = \Delta_{\boldsymbol{r}^-}^{-1}(C^{-+} + C^{--}\Psi^{-+}). \tag{2.2}$$

If $d \ge 0$, i.e., the mean drift of the Markov additive process is not negative, Ψ^{-+} and $Q^{(+)}$ are stochastic and non-defective matrices, respectively. Otherwise, Ψ^{-+} and $Q^{(+)}$ are substochastic and defective matrices, respectively.

We next introduce a dual process $(\hat{X}(t), \hat{M}(t))$, which is defined by

$$(\hat{X}(t), \hat{M}(t)) = (-X(-t), M(-t)), \tag{2.3}$$

where $\hat{M}(t)$ is the continuous time Markov chain with transition rate matrix $\hat{C} \equiv \Delta_{\boldsymbol{\pi}}^{-1} C^{\mathrm{T}} \Delta_{\boldsymbol{\pi}}$. We partition $\hat{C}$ according to $\mathcal{S}^-$ and $\mathcal{S}^+$ as

$$\hat{C} = \begin{pmatrix} \hat{C}^{--} & \hat{C}^{-+} \\ \hat{C}^{+-} & \hat{C}^{++} \end{pmatrix}.$$

Let $\hat{\Psi}_{-\infty}^{\bullet +}(0, x)$ be the hitting probability for the dual process with the same interpretation as $\Psi_{-\infty}^{\bullet +}(0, x)$ for the original Markov additive process. Similarly to Proposition 2.1, we have the following result.

Corollary 2.1 For $x > 0$, we have

$$\hat{\Psi}^{\bullet +}_{-\infty}(0,x) = \begin{pmatrix} \hat{\Psi}^{-+} \exp(x\hat{Q}^{(+)}) \\ \exp(x\hat{Q}^{(+)}) \end{pmatrix},$$

where $\hat{\Psi}^{-+}$ and $\hat{Q}^{(+)}$ are the minimal solutions of the following equations.

$$\hat{Q}^{(+)} = \Delta^{-1}_{\boldsymbol{r}^+}(\hat{C}^{++} + \hat{C}^{+-}\hat{\Psi}^{-+}), \quad \hat{\Psi}^{-+}\hat{Q}^{(+)} = \Delta^{-1}_{\boldsymbol{r}^-}(\hat{C}^{-+} + \hat{C}^{--}\hat{\Psi}^{-+}).$$

If $d \geq 0$, $\hat{\Psi}^{-+}$ and $\hat{Q}^{(+)}$ are stochastic and non-defective matrices, respectively. Otherwise, $\hat{\Psi}^{-+}$ and $\hat{Q}^{(+)}$ are substochastic and defective matrices, respectively.

We show that $\Phi^{+\bullet}_0(0,x)$ is obtained from the hitting probabilities of the dual process. This result may be considered as a fluid version of Proposition 2.13 of chapter XI in [1].

Lemma 2.1 For $x > 0$, we have

$$\Phi^{+\bullet}_0(0,x) = \Delta^{-1}_{\boldsymbol{\pi}^+}\Delta^{-1}_{\boldsymbol{r}^+}\left(\int_0^x (\hat{\Psi}^{\bullet +}_{-\infty}(0,y))^{\mathrm{T}} dy\right)\Delta_{\boldsymbol{\pi}}.$$

PROOF. For $x > 0$, $i \in \mathcal{S}^+$ and $j \in \mathcal{S}$, we have

$$\begin{aligned}
&\pi_i[\Phi^{+\bullet}_0(0,x)]_{ij} \\
&= \pi_i E\left[\int_0^{\tau_{X(0)}} 1(X(t) - X(0) \leq x, M(t) = j)dt \,\middle|\, M(0) = i\right] \\
&= \int_0^\infty P\Big(M(t) = j, X(t) - X(0) \leq x, X(u) - X(0) > 0, u \in (0,t), M(0) = i\Big)dt \\
&= \int_0^\infty P\Big(M(0) = j, X(0) - X(-t) \leq x, X(u-t) - X(-t) > 0, u \in (0,t), M(-t) = i\Big)dt \\
&= \int_0^\infty P\Big(\hat{M}(0) = j, \hat{M}(t) = i, \hat{X}(t) - \hat{X}(0) \leq x, \hat{X}(t) - \hat{X}(u) > 0, u \in (0,t)\Big)dt \\
&= E\left[\int_0^\infty 1\left(\hat{M}(0) = j, \hat{M}(t) = i, \hat{X}(t) - \hat{X}(0) \leq x, t = \hat{\tau}_{\hat{X}(t)-\hat{X}(0)}\right)dt\right],
\end{aligned}$$

where we have shifted the time by $-t$ in the third equation, and $\hat{\tau}_a \equiv \inf\{t > 0 | \hat{X}(t) = a\}$ for $a \in \mathbb{R}$. Noting that $\hat{M}(t)$ also has the stationary distribution $\boldsymbol{\pi}$, we have

$$\begin{aligned}
&\pi_i[\Phi^{+\bullet}_0(0,x)]_{ij} \\
&= \pi_j \frac{1}{r_i} E\left[\int_0^\infty 1\left(\hat{M}(t) = i, \hat{X}(t) - \hat{X}(0) \leq x, t = \hat{\tau}_{\hat{X}(t)-\hat{X}(0)}\right) r_i dt \,\middle|\, \hat{M}(0) = j\right] \\
&= \pi_j \frac{1}{r_i} \int_0^x P\Big(\hat{M}(\hat{\tau}_y) = i \,\Big|\, \hat{M}(0) = j\Big)dy \\
&= \pi_j \frac{1}{r_i} \int_0^x [\hat{\Psi}^{\bullet +}_{-\infty}(0,y)]_{ji} dy,
\end{aligned}$$

where the second equation follows since $r_{M(t)}dt = dX(t)$ except for time epoch that $X(t)$ changes, which completes the proof of the lemma. □

Lemma 2.1 shows that the occupation measure is differentiable. Denoting the density of $\Phi_0^{+\bullet}(0, x)$ by $\varphi_0^{+\bullet}(0, x)$, we have the following result from Corollary 2.1 and Lemma 2.1, which implies that the density is given by the hitting probabilities of the dual process.

Corollary 2.2 For $x > 0$, we have

$$\varphi_0^{+\bullet}(0, x) = \Delta_{\boldsymbol{\pi}^+}^{-1} \Delta_{\boldsymbol{r}^+}^{-1} \begin{pmatrix} \hat{\Psi}^{-+} \exp(x\hat{Q}^{(+)}) \\ \exp(y\hat{Q}^{(+)}) \end{pmatrix}^{\mathrm{T}} \Delta_{\boldsymbol{\pi}}. \tag{2.4}$$

It is intuitively clear that $\varphi_0^{+\bullet}(0, x)\Delta_{|\boldsymbol{r}|}$ is the expected number of level crossing times at level x before returning to the initial level 0. In fact, we have the following result, which is proved in Appendix A.

Lemma 2.2 For $i \in \mathcal{S}^+$ and $j \in \mathcal{S}$, the (i, j)-th element of $\varphi_0^{+\bullet}(0, x)\Delta_{|\boldsymbol{r}|}$ has the following expression.

$$[\varphi_0^{+\bullet}(0, x)\Delta_{|\boldsymbol{r}|}]_{ij} = E\left[\sum_{0 \le t < \tau_0} 1(X(t) = x, M(t) = j) \middle| X(0) = 0, M(0) = i \right].$$

We finally consider the stationary measure of the background process for the case of $d = 0$, which will be used in Section 5. For $n \ge 0$, let γ_n be the n-th time that the Markov additive process crosses the initial level, where $\gamma_0 \equiv 0$. We note that $\gamma_{n+1} - \gamma_n$ is finite with probability one for $d = 0$. Then, $\{M(\gamma_n); n \ge 0\}$ is the embedded Markov chain with state space $\mathcal{S}$, and has the transition probability matrix

$$U = \begin{pmatrix} O & \Psi^{-+} \\ \Psi^{+-} & O \end{pmatrix}.$$

Let $\boldsymbol{\xi} = (\boldsymbol{\xi}^-, \boldsymbol{\xi}^+)$ be the stationary distribution of this Markov chain. Then, it is given as follows. Since the proof of this lemma is just technical, we defer it to Appendix B.

Lemma 2.3 For $d = 0$, we have

$$\boldsymbol{\xi}^+ = \frac{1}{\boldsymbol{\pi}^+\boldsymbol{r}^+ - \boldsymbol{\pi}^-\boldsymbol{r}^-}(\boldsymbol{r}^+)^{\mathrm{T}}\Delta_{\boldsymbol{\pi}^+}, \quad \boldsymbol{\xi}^- = \frac{-1}{\boldsymbol{\pi}^+\boldsymbol{r}^+ - \boldsymbol{\pi}^-\boldsymbol{r}^-}(\boldsymbol{r}^-)^{\mathrm{T}}\Delta_{\boldsymbol{\pi}^-}.$$

3 Fluid queue with a finite buffer

In this section, we put two boundaries at levels 0 and b to the Markov additive process in the last section so that the level process stays in $[0, b]$, where $b > 0$. To describe its behavior at the two boundaries, we use Markov chains $\underline{M}(t)$ and $\overline{M}(t)$ with the transition rate matrices

$$\begin{pmatrix} \underline{C}^{--} & \underline{C}^{-+} \\ O & O \end{pmatrix}, \quad \begin{pmatrix} O & O \\ \overline{C}^{+-} & \overline{C}^{++} \end{pmatrix},$$

respectively. We define the reflected additive process $(Y^{(b)}(t), J^{(b)}(t))$ in the following way.

(i) While $Y^{(b)}(t)$ remains in $(0, b)$, $(Y^{(b)}(t), J^{(b)}(t))$ has the same transition structure as the Markov additive process $(X(t), M(t))$, i.e.,

$$\frac{d}{dt}Y^{(b)}(t) = r(J^{(b)}(t)), \quad J^{(b)}(t) = M(t).$$

(ii) Let σ_0 be the first time when the level process hits level 0. Then, for $t \geq \sigma_0$,

$$Y^{(b)}(t) = 0, \quad J^{(b)}(t) = \underline{M}(t)$$

with $\underline{M}(\sigma_0) = M(\sigma_0-)$ as long as $r(J^{(b)}(t)) < 0$. Let $\sigma_+ = \min\{u > \sigma_0 | r(J^{(b)}(u)) > 0\}$, then $(Y^{(b)}(t), J^{(b)}(t))$ restarts subject to the same transition structure as (i) with $M(\sigma_+) = \underline{M}(\sigma_+-)$ for $t \geq \sigma_+$.

(iii) Similarly to (ii), let σ_b be the first time when the level process hits level b. Then, for $t \geq \sigma_b$,

$$Y^{(b)}(t) = b, \quad J^{(b)}(t) = \overline{M}(t)$$

with $\overline{M}(\sigma_b) = M(\sigma_b-)$ as long as $r(J^{(b)}(t)) > 0$. Let $\sigma_- = \min\{u > \sigma_b | r(J^{(b)}(u)) < 0\}$, then $(Y^{(b)}(t), J^{(b)}(t))$ restarts subject to the same transition structure as (i) with $M(\sigma_-) = \overline{M}(\sigma_--)$ for $t \geq \sigma_-$.

From (i), (ii) and (iii), $(Y^{(b)}(t), J^{(b)}(t))$ is a continuous time Markov chain with state space $(\{0\} \times \mathcal{S}^-) \cup ((0, b) \times \mathcal{S}) \cup (\{b\} \times \mathcal{S}^+)$. In applications, the net flow rate r may be obtained as

$$r(i) = r_{\text{in}}(i) - r_{\text{out}}(i), \quad i \in \mathcal{S},$$

where r_{in} and r_{out} are nonnegative valued functions on $\mathcal{S}$, and represent the fluid input and output rates, respectively. Then, $(Y^{(b)}(t), J^{(b)}(t))$ is referred to as a Markov modulated fluid queue with a finite buffer of size b (finite fluid queue, for short).

The loss rate of the finite fluid queue is defined as the expected amount of lost fluid per unit time when the buffer is full. Namely,

Definition 3.1 We define the loss rate of the finite fluid queue by

$$\ell^{(b)}_{\text{Loss}} = E\left[\int_0^1 r_{\text{in}}(J^{(b)}(u))1(Y^{(b)}(u) = b, J^{(b)}(u) = j)du\right],$$

where the expectation is taken under the stationary distribution of $(Y^{(b)}(t), J^{(b)}(t))$.

We tentatively assume the existence of the stationary distribution for the finite fluid queue. Let $\overline{\boldsymbol{p}}^{(b)}$ and $\underline{\boldsymbol{p}}^{(b)}$ be the $|\mathcal{S}^+|$ and $|\mathcal{S}^-|$-dimensional stationary probability vectors of the background process when the buffer is full and empty, respectively. Then, it is easy to see that the loss rate is given by

$$\ell^{(b)}_{\text{Loss}} = \overline{\boldsymbol{p}}^{(b)}\boldsymbol{r}^+_{\text{in}},$$

where $\boldsymbol{r}_{\rm in}^{+} \equiv (r_{\rm in}(i); i \in \mathcal{S}^{+})$. To get the asymptotic behavior of the loss rate, we consider the tail asymptotics of $\overline{\boldsymbol{p}}^{(b)}$ as b gets large.

To this end, we first construct the stationary measure of the finite fluid queue in the following way. By censoring out all the off boundary states, the transition rate matrix of the background process at the two boundaries, levels 0 and b, is given by

$$\begin{pmatrix} \underline{C}^{--} + \underline{C}^{-+}\Psi_b^{+-}(0,0) & \underline{C}^{-+}\Psi_0^{++}(0,b) \\ \overline{C}^{+-}\Psi_b^{--}(b,0) & \overline{C}^{++} + \overline{C}^{+-}\Psi_0^{-+}(b,b) \end{pmatrix},$$

which is non-defective since $\Psi_b^{+-}(0,0)\mathbf{1}+\Psi_0^{++}(0,b)\mathbf{1} = \mathbf{1}$ and $\Psi_b^{--}(b,0)\mathbf{1}+\Psi_0^{-+}(b,b)\mathbf{1} = \mathbf{1}$. Then, there exist the stationary measures of level 0 and b which is denoted by $\underline{\boldsymbol{s}}^{(b)}$ and $\overline{\boldsymbol{s}}^{(b)}$, respectively. They are uniquely determined (up to multiplicative constant) by

$$\underline{\boldsymbol{s}}^{(b)}(\underline{C}^{--} + \underline{C}^{-+}\Psi_b^{+-}(0,0)) + \overline{\boldsymbol{s}}^{(b)}\overline{C}^{+-}\Psi_b^{--}(b,0) = \mathbf{0}, \tag{3.1}$$

$$\underline{\boldsymbol{s}}^{(b)}\underline{C}^{-+}\Psi_0^{++}(0,b) + \overline{\boldsymbol{s}}^{(b)}(\overline{C}^{++} + \overline{C}^{+-}\Psi_0^{-+}(b,b)) = \mathbf{0}. \tag{3.2}$$

For $0 < x < b$, let

$$\boldsymbol{h}^{(b)}(x) = \underline{\boldsymbol{s}}^{(b)}\underline{C}^{-+}\Phi_{0b}^{+\bullet}(0,x) + \overline{\boldsymbol{s}}^{(b)}\overline{C}^{+-}\Phi_{0b}^{-\bullet}(b,x). \tag{3.3}$$

We can see that for $0 < x < b$ and $j \in \mathcal{S}$, $[\boldsymbol{h}^{(b)}(x)]_j$ is the mean sojourn time in subset $(0, x] \times \{j\}$ between two successive visiting time at the two boundaries, starting from the boundaries according to the stationary measure $(\underline{\boldsymbol{s}}^{(b)}, \overline{\boldsymbol{s}}^{(b)})$. Hence, we have the following result.

Lemma 3.1 $\underline{\boldsymbol{s}}^{(b)}$, $\overline{\boldsymbol{s}}^{(b)}$ and $\boldsymbol{h}^{(b)}(x)$ constitute the stationary measure μ of the finite fluid queue in such a way that

$$\mu(j,B) = [\underline{\boldsymbol{s}}^{(b)}]_j 1(j \in \mathcal{S}^{-}, 0 \in B) + [\overline{\boldsymbol{s}}^{(b)}]_j 1(j \in \mathcal{S}^{+}, b \in B) + \int_B [\boldsymbol{h}^{(b)}(dx)]_j$$

for $j \in \mathcal{S}$ and Borel set B on $[0, b]$.

The stationary probability vectors of levels 0 and b are given by normalizing their stationary measures, that is,

$$\underline{\boldsymbol{p}}^{(b)} = \frac{\underline{\boldsymbol{s}}^{(b)}}{c_b}, \quad \overline{\boldsymbol{p}}^{(b)} = \frac{\overline{\boldsymbol{s}}^{(b)}}{c_b}, \tag{3.4}$$

where $c_b \equiv \underline{\boldsymbol{s}}^{(b)}\mathbf{1} + \boldsymbol{h}^{(b)}(b-)\mathbf{1} + \overline{\boldsymbol{s}}^{(b)}\mathbf{1}$.

Similarly to Lemma 2.1, $\Phi_{0b}^{+\bullet}(0,x)$ and $\Phi_{0b}^{-\bullet}(b,x)$ in the right hand side of (3.3) are given by hitting probabilities as follows, which implies that $\boldsymbol{h}^{(b)}(x)$ is differentiable. The proof of the following lemma is deferred to Appendix C since it is similar to the ones demonstrated in Lemma 2.1.

Lemma 3.2 For $x \in (0, b)$ and $j \in \mathcal{S}$, we have

$$[\Phi_{0b}^{+\bullet}(0,x)]_{ij} = \frac{\pi_j}{r_i \pi_i} \int_0^x P(\hat{M}(\hat{\tau}_y) = i, \hat{\tau}_y \le \hat{\eta}^{(b)} | \hat{M}(0) = j) dy, \quad i \in \mathcal{S}^{+},$$

$$[\Phi_{0b}^{-\bullet}(b,x)]_{ij} = \frac{\pi_j}{|r_i| \pi_i} \int_{-b}^{-(b-x)} P(\hat{M}(\hat{\tau}_y) = i, \hat{\tau}_y \le \hat{\eta}^{(b)} | \hat{M}(0) = j) dy, \quad i \in \mathcal{S}^{-},$$

where $\hat{\eta}^{(b)} \equiv \min\{t > 0 | \max_{u \in (0,t)} \hat{X}(u) - \min_{u \in (0,t)} \hat{X}(u) = b\}$.

Denote the derivatives of $\boldsymbol{h}^{(b)}(x)$, $\Phi_{0b}^{+\bullet}(0,x)$ and $\Phi_{0b}^{-\bullet}(b,x)$ by $\boldsymbol{\nu}^{(b)}(x) \equiv (\boldsymbol{\nu}^{(b)-}(x), \boldsymbol{\nu}^{(b)+}(x))$, $\varphi_{0b}^{+\bullet}(0,x)$ and $\varphi_{0b}^{-\bullet}(b,x)$, respectively. From (3.3) and Lemma 3.2, we have the following result, which implies that the stationary density of the finite fluid queue is given by hitting probabilities of the dual process $(\hat{X}(t), \hat{M}(t))$ in (2.3).

Theorem 3.1 For a Markov modulated fluid queue with a finite buffer of size b, we have

$$\boldsymbol{\nu}^{(b)}(x) = \underline{\boldsymbol{s}}^{(b)}\underline{C}^{-+}\varphi_{0b}^{+\bullet}(0,x) + \overline{\boldsymbol{s}}^{(b)}\overline{C}^{+-}\varphi_{0b}^{-\bullet}(b,x), \quad x \in (0,b), \tag{3.5}$$

where $\varphi_{0b}^{+\bullet}(0,x)$ and $\varphi_{0b}^{-\bullet}(b,x)$ are given by

$$[\varphi_{0b}^{+\bullet}(0,x)]_{ij} = \frac{\pi_j}{r_i\pi_i}P(\hat{M}(\hat{\tau}_x) = i, \hat{\tau}_x \le \hat{\eta}^{(b)} | \hat{M}(0) = j), \quad i \in \mathcal{S}^+, j \in \mathcal{S}, \tag{3.6}$$

$$[\varphi_{0b}^{-\bullet}(b,x)]_{ij} = \frac{\pi_j}{|r_i|\pi_i}P(\hat{M}(\hat{\tau}_{-(b-x)}) = i, \hat{\tau}_{-(b-x)} \le \hat{\eta}^{(b)} | \hat{M}(0) = j), \quad i \in \mathcal{S}^-, j \in \mathcal{S}.$$

Remark 3.1 For $d \neq 0$, (3.5) can be rewritten in the form of combination of matrix exponentials (see Appendix D), which is equivalent to the result in Theorem 4.4 of [5] when $\boldsymbol{r}^+ = \mathbf{1}$ and $\boldsymbol{r}^- = -\mathbf{1}$. On the other hand, as noted in Lemma 4.2 of [5], (3.5) does not have such matrix exponential formula for $d = 0$. The representation in (3.5) is valid including the case of $d = 0$.

Remark 3.2 Taking the limit as $b \to \infty$ in (3.6), we obtain (2.4).

From (3.2), (3.3) and Lemma 3.2, we can see that the stationary measure of level b has the following form, which is useful when we consider the tail asymptotics of the loss rate for $d = 0$. The derivation of this formula is given in Appendix E.

Lemma 3.3 We have

$$\overline{\boldsymbol{s}}^{(b)} = \boldsymbol{\nu}^{(b)+}(b-)\Delta_{\boldsymbol{r}^+}(-\overline{C}^{++})^{-1}. \tag{3.7}$$

4 Fluid queue with an infinite buffer

By removing the boundary at level b of the finite fluid queue in the last section, we obtain a Markov modulated fluid queue with an infinite buffer (infinite fluid queue, for short). Denote this infinite fluid queue by $(Y(t), J(t))$, which is a continuous time Markov chain with state space $(\{0\} \times \mathcal{S}^-) \cup ((0,\infty) \times \mathcal{S})$.

We assume $d \le 0$ so that the recurrence time to level 0 is finite with probability one. Similarly to the finite fluid queue, the infinite fluid queue has the stationary measure as follows. By censoring out all the off boundaries, the transition rate matrix of the background process at level 0 is given by $\underline{C}^{--} + \underline{C}^{-+}\Psi^{+-}$, which is non-defective since Ψ^{+-} is a stochastic matrix when $d \le 0$. Then, there exists the stationary measure $\underline{\boldsymbol{s}}$ of level 0, which is uniquely (up to multiplicative constant) determined by

$$\underline{\boldsymbol{s}}(\underline{C}^{--} + \underline{C}^{-+}\Psi^{+-}) = \mathbf{0}.$$

For $x > 0$, let $\boldsymbol{h}(x)$ be the $|\mathbb{S}|$-dimensional row vector given by

$$\boldsymbol{h}(x) = \underline{\boldsymbol{s}}\, \underline{C}^{-+} \Phi_0^{+\bullet}(0, x).$$

Similarly to Lemma 3.1, $\underline{\boldsymbol{s}}$ and $\boldsymbol{h}(x)$ constitute the stationary measure of the infinite fluid queue.

From Lemma 2.1 and Corollary 2.2, $\boldsymbol{h}(x)$ has the density $\boldsymbol{\nu}(x)$.

Lemma 4.1 For $d \leq 0$, we have

$$\boldsymbol{\nu}(x) = \underline{\boldsymbol{s}}\, \underline{C}^{-+} \Delta_{\boldsymbol{\pi}^+}^{-1} \Delta_{\boldsymbol{r}^+}^{-1} \begin{pmatrix} \hat{\Psi}^{-+} \exp(x\hat{Q}^{(+)}) \\ \exp(x\hat{Q}^{(+)}) \end{pmatrix}^{\mathrm{T}} \Delta_{\boldsymbol{\pi}}, \quad x > 0.$$

From Lemma 4.1, we obtain the following formula, which relates the infinite fluid queue to the Markov additive process when $d = 0$.

Lemma 4.2 For $d = 0$, we have

$$\lim_{b \to \infty} \frac{\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|} = \boldsymbol{\xi}.$$

Proof. Since $\hat{Q}^{(+)}$ has the stationary distribution $\hat{\boldsymbol{\kappa}}^+$ for $d = 0$ (see also Appendix B), we have

$$\begin{aligned} \lim_{b \to \infty} \frac{\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|} &= \frac{\underline{\boldsymbol{s}}\, \underline{C}^{-+} \Delta_{\boldsymbol{\pi}^+}^{-1} \Delta_{\boldsymbol{r}^+}^{-1} \left((\hat{\Psi}^{-+}\mathbf{1}^+\hat{\boldsymbol{\kappa}}^+)^{\mathrm{T}}, (\mathbf{1}^+\hat{\boldsymbol{\kappa}}^+)^{\mathrm{T}} \right) \Delta_{\boldsymbol{\pi}} \Delta_{|\boldsymbol{r}|}}{\underline{\boldsymbol{s}}\, \underline{C}^{-+} \Delta_{\boldsymbol{\pi}^+}^{-1} \Delta_{\boldsymbol{r}^+}^{-1} \left((\hat{\Psi}^{-+}\mathbf{1}^+\hat{\boldsymbol{\kappa}}^+)^{\mathrm{T}}, (\mathbf{1}^+\hat{\boldsymbol{\kappa}}^+)^{\mathrm{T}} \right) \Delta_{\boldsymbol{\pi}} \Delta_{|\boldsymbol{r}|} \mathbf{1}} \\ &= \frac{\{\underline{\boldsymbol{s}}\, \underline{C}^{-+} \Delta_{\boldsymbol{\pi}^+}^{-1} \Delta_{\boldsymbol{r}^+}^{-1} (\hat{\boldsymbol{\kappa}}^+)^{\mathrm{T}}\} \boldsymbol{\pi} \Delta_{|\boldsymbol{r}|}}{\{\underline{\boldsymbol{s}}\, \underline{C}^{-+} \Delta_{\boldsymbol{\pi}^+}^{-1} \Delta_{\boldsymbol{r}^+}^{-1} (\hat{\boldsymbol{\kappa}}^+)^{\mathrm{T}}\} \boldsymbol{\pi} |\boldsymbol{r}|} \\ &= \boldsymbol{\xi}. \end{aligned}$$

□

Instead of level b, if we remove the boundary at level 0 of the finite fluid queue, we obtain another infinite fluid queue by changing the sign of the level process. Denote this infinite fluid queue and its density by $(\tilde{Y}(t), \tilde{J}(t))$ and $\tilde{\boldsymbol{\nu}}(x)$, respectively. Similarly to the preceding argument, we obtain the following results. Assuming that $d \geq 0$, there exists the stationary measure $\overline{\boldsymbol{s}}$ such that

$$\overline{\boldsymbol{s}}(\overline{C}^{++} + \overline{C}^{+-} \Psi^{-+}) = \mathbf{0}.$$

Then, we have the following lemmas.

Lemma 4.3 For $d \geq 0$, we have

$$\tilde{\boldsymbol{\nu}}(x) = \overline{\boldsymbol{s}}\, \overline{C}^{+-} \Delta_{\boldsymbol{\pi}^-}^{-1} \Delta_{|\boldsymbol{r}^-|}^{-1} \begin{pmatrix} \exp(x\tilde{Q}^{(-)}) \\ \tilde{\Psi}^{+-} \exp(x\tilde{Q}^{(-)}) \end{pmatrix}^{\mathrm{T}} \Delta_{\boldsymbol{\pi}}, \quad x > 0,$$

where $\tilde{\Psi}^{+-}$ and $\tilde{Q}^{(-)}$ are the minimal solutions of the following equations.

$$\tilde{\Psi}^{+-}\tilde{Q}^{(-)} = (-\Delta_{\boldsymbol{r}^+})^{-1}(\hat{C}^{+-} + \hat{C}^{++}\tilde{\Psi}^{+-}), \quad \tilde{Q}^{(-)} = \Delta_{|\boldsymbol{r}^-|}^{-1}(\hat{C}^{--} + \hat{C}^{-+}\tilde{\Psi}^{+-}). \tag{4.1}$$

Lemma 4.4 For $d = 0$, we have

$$\lim_{b\to\infty} \frac{\tilde{\boldsymbol{\nu}}(b/2)\Delta_{|\boldsymbol{r}|}}{\|\tilde{\boldsymbol{\nu}}(b/2)\Delta_{|\boldsymbol{r}|}\|} = \boldsymbol{\xi}.$$

5 Asymptotic behavior of loss rate

We obtain asymptotic behavior of the loss rate of the finite fluid queue as the buffer size gets large. As noted in Section 3, it is sufficient to derive the tail asymptotics of $\overline{\boldsymbol{p}}^{(b)}$. We first consider the case of $d \neq 0$. In this case, we can apply Theorem 1 in [13], where a finite fluid queue may have downward jumps. However, the prefactor of the exponential term is more complicated because of the jump structure. We here derived a simpler expression. Since the proof is technical, it is deferred to Appendix F.

Theorem 5.1 (i) For $d < 0$, we have

$$\lim_{b\to\infty} e^{\alpha b}\, \overline{\boldsymbol{p}}^{(b)} = \underline{\boldsymbol{p}}\underline{C}^{-+}(I - \Psi^{+-}\Psi^{-+})\boldsymbol{g}^{+}\boldsymbol{\eta}^{+}\Delta_{\boldsymbol{g}^{+}}^{-1}(-\overline{C}^{++} - \overline{C}^{+-}\Psi^{-+})^{-1},$$

where $-\alpha$ is the Perron Frobeneus eigenvalue of the defective matrix $Q^{(+)}$ in (2.2) with a corresponding right eigenvector $\boldsymbol{g}^{+}$, $\boldsymbol{\eta}^{+}$ is the stationary distribution of the stochastic matrix $\Delta_{\boldsymbol{g}^{+}}^{-1}(\alpha I + Q^{(+)})\Delta_{\boldsymbol{g}^{+}}$, and the row vector $\underline{\boldsymbol{p}}$ is a normalized $\underline{\boldsymbol{s}}$ such that

$$\underline{\boldsymbol{p}}\mathbf{1} + \underline{\boldsymbol{p}}\underline{C}^{-+}\Delta_{\boldsymbol{\pi}^{+}}^{-1}\Delta_{\boldsymbol{r}^{+}}^{-1}\begin{pmatrix} \hat{\Psi}^{-+}(-\hat{Q}^{(+)})^{-1} \\ (-\hat{Q}^{(+)})^{-1} \end{pmatrix}^{\mathrm{T}}\Delta_{\boldsymbol{\pi}}\mathbf{1} = 1.$$

(ii) For $d > 0$, $\overline{\boldsymbol{p}}^{(b)}$ converges to $\overline{\boldsymbol{p}}$ as b gets large, where $\overline{\boldsymbol{p}}$ is a normalized $\overline{\boldsymbol{s}}$ such that

$$\overline{\boldsymbol{p}}\mathbf{1} + \overline{\boldsymbol{p}}\overline{C}^{+-}\Delta_{\boldsymbol{\pi}^{-}}^{-1}\Delta_{|\boldsymbol{r}^{-}|}^{-1}\begin{pmatrix} (-\tilde{Q}^{(-)})^{-1} \\ \tilde{\Psi}^{+-}(-\tilde{Q}^{(-)})^{-1} \end{pmatrix}^{\mathrm{T}}\Delta_{\boldsymbol{\pi}}\mathbf{1} = 1.$$

The next result shows that $\overline{\boldsymbol{p}}^{(b)}$ decays proportionally to $1/b$ when $d = 0$.

Theorem 5.2 For $d = 0$, we have

$$\lim_{b\to\infty} b\, \overline{\boldsymbol{p}}^{(b)} = \frac{1}{\overline{\boldsymbol{s}}\overline{C}^{+-}\Delta_{\boldsymbol{\pi}^{-}}^{-1}\Delta_{|\boldsymbol{r}^{-}|}^{-1}(\tilde{\boldsymbol{\beta}}^{-})^{\mathrm{T}}}\tilde{\boldsymbol{\nu}}^{+}(0+)\Delta_{\boldsymbol{r}^{+}}(-\overline{C}^{++})^{-1},$$

where $\tilde{\boldsymbol{\nu}}^{+}(0+) = \overline{\boldsymbol{s}}\overline{C}^{+-}\Delta_{\boldsymbol{\pi}^{-}}^{-1}\Delta_{|\boldsymbol{r}^{-}|}^{-1}(\tilde{\Psi}^{+-})^{\mathrm{T}}\Delta_{\boldsymbol{\pi}^{+}}$ and $\tilde{\Psi}^{+-}$ is given by (4.1).

From Theorem 5.1 and Theorem 5.2, we obtain asymptotic behavior of the loss rate of the finite fluid queue as follows.

Corollary 5.1 For $d < 0$, we have

$$\lim_{b\to\infty} e^{\alpha b}\, \ell_{\mathrm{Loss}}^{(b)} = \underline{\boldsymbol{p}}\,\underline{C}^{-+}(I - \Psi^{+-}\Psi^{-+})\boldsymbol{g}^{+}\boldsymbol{\eta}^{+}\Delta_{\boldsymbol{g}^{+}}^{-1}(-\overline{C}^{++} - \overline{C}^{+-}\Psi^{-+})^{-1}\boldsymbol{r}_{\mathrm{in}}^{+}.$$

For $d > 0$, we have

$$\lim_{b\to\infty} \ell^{(b)}_{\mathrm{Loss}} = \overline{\boldsymbol{p}}\boldsymbol{r}^{+}_{\mathrm{in}}.$$

For $d = 0$, we have

$$\lim_{b\to\infty} b\, \ell^{(b)}_{\mathrm{Loss}} = \frac{1}{\overline{\boldsymbol{s}}\overline{C}^{+-}\Delta^{-1}_{\boldsymbol{\pi}^-}\Delta^{-1}_{|\boldsymbol{r}^-|}(\tilde{\boldsymbol{\beta}}^-)^{\mathrm{T}}}\tilde{\boldsymbol{\nu}}^{+}(0+)\Delta_{\boldsymbol{r}^+}(-\overline{C}^{++})^{-1}\boldsymbol{r}^{+}_{\mathrm{in}}.$$

(Proof of Theorem 5.2) We apply a similar technique used in [8]. To this end, we prepare the following lemmas, whose proofs are deferred to Appendix G. We first note that from Lemma 2.2 and (3.5), the j-th element of $\boldsymbol{\nu}^{(b)}(x)\Delta_{|\boldsymbol{r}|}$ equals (up to multiplicative constant) the expected number of crossing times at level x with background state j before the first recurrence time to level $b/2$ with a fixed background state i^*, starting from level $b/2$ with background state i^*, that is,

$$[\boldsymbol{\nu}^{(b)}(x)\Delta_{|\boldsymbol{r}|}]_j = E\left[\sum_{0\le t<r^{(b)}_{b/2}(i^*)} 1(Y^{(b)}(t) = x, J^{(b)}(t) = j)\middle| Y^{(b)}(0) = b/2, J^{(b)}(0) = i^*\right] \quad (5.1)$$

for $j \in \mathcal{S}$, where $\tau^{(b)}_{b/2}(i^*) \equiv \inf\{t > 0|(Y^{(b)}(t), J^{(b)}(t)) = (b/2, i^*)\}$. Then, we can see that

$$\frac{[\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}]_j}{[\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}]_i} = E\left[\sum_{0\le t<\tau^{(b)}_{b/2}(i)} 1\left((Y^{(b)}(t), J^{(b)}(t)) = (b/2, j)\right)\middle|(Y^{(b)}(0), J^{(b)}(0)) = (b/2, i)\right] \quad (5.2)$$

for $i, j \in \mathcal{S}$. Since the boundary effect on the right side of (5.2) disappears as b gets large, we obtain the following lemma.

Lemma 5.1

$$\lim_{b\to\infty} \frac{\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}}{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|} = \boldsymbol{\xi}.$$

By Lemma 4.2 and Lemma 5.1, we obtain the following lemma.

Lemma 5.2

$$\lim_{b\to\infty} \max_{x\in(0,b/2),i\in\mathcal{S}} \max\left(\frac{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|}{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}\frac{[\boldsymbol{\nu}^{(b)}(x)]_i}{[\boldsymbol{\nu}(x)]_i}, \frac{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|}\frac{[\boldsymbol{\nu}(x)]_i}{[\boldsymbol{\nu}^{(b)}(x)]_i}\right) = 1.$$

Similarly to Lemma 5.2, we have

Corollary 5.2

$$\lim_{b\to\infty} \max_{x\in(0,b/2),i\in\mathcal{S}} \max\left(\frac{\|\tilde{\boldsymbol{\nu}}(b/2)\Delta_{|\boldsymbol{r}|}\|}{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}\frac{[\boldsymbol{\nu}^{(b)}(b-x)]_i}{[\tilde{\boldsymbol{\nu}}(x)]_i}, \frac{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}{\|\tilde{\boldsymbol{\nu}}(b/2)\Delta_{|\boldsymbol{r}|}\|}\frac{[\tilde{\boldsymbol{\nu}}(x)]_i}{[\boldsymbol{\nu}^{(b)}(b-x)]_i}\right) = 1.$$

From Lemma 5.2, Corollary 5.2 and Lemma 3.1, we have

Lemma 5.3

$$\lim_{b\to\infty}\left(\frac{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}{c_b}\int_0^{b/2}\frac{\|\boldsymbol{\nu}(x)\|}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|}dx+\frac{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}{c_b}\int_0^{b/2}\frac{\|\tilde{\boldsymbol{\nu}}(x)\|}{\|\tilde{\boldsymbol{\nu}}(b/2)\Delta_{|\boldsymbol{r}|}\|}dx\right)=1.$$

We return to the proof of Theorem 5.2. From the proof of Lemma 4.2, we have

$$\lim_{b\to\infty}\frac{\boldsymbol{\nu}(b/2)}{\|\boldsymbol{\nu}(b/2)\|}=\boldsymbol{\pi},$$

which implies that for arbitrary $\epsilon>0$, there exists $\delta>0$ such that

$$1-\epsilon<\frac{\|\boldsymbol{\nu}(x)\|}{\|\boldsymbol{\nu}(b/2)\|}<1+\epsilon,\quad x,\frac{b}{2}\geq\delta.$$

Then we have

$$\begin{aligned}&\frac{\int_0^{b/2}\frac{\|\boldsymbol{\nu}(x)\|}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|}dx}{b/2}=\frac{h(\delta)+\frac{\|\boldsymbol{\nu}(b/2)\|}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|}\int_\delta^{b/2}\frac{\|\boldsymbol{\nu}(x)\|}{\|\boldsymbol{\nu}(b/2)\|}dx}{b/2}\\&\in\left(\frac{h(\delta)}{b/2}+\frac{\|\boldsymbol{\nu}(b/2)\|}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|}\left(1-\frac{2\delta}{b}\right)(1-\epsilon),\frac{h(\delta)}{b/2}+\frac{\|\boldsymbol{\nu}(b/2)\|}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|}\left(1-\frac{2\delta}{b}\right)(1+\epsilon)\right),\end{aligned}\tag{5.3}$$

where $h(\delta)=\int_0^\delta\frac{\|\boldsymbol{\nu}(x)\|}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|}dx$. From the proof of Lemma 4.2 and taking limit of (5.3), we have

$$\lim_{b\to\infty}\frac{\int_0^{b/2}\frac{\|\boldsymbol{\nu}(x)\|}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|}dx}{b/2}\in\left(\frac{1-\epsilon}{\boldsymbol{\pi}|\boldsymbol{r}|},\frac{1+\epsilon}{\boldsymbol{\pi}|\boldsymbol{r}|}\right).$$

Since ϵ is arbitrary, we have

$$\lim_{b\to\infty}\frac{\int_0^{b/2}\frac{\|\boldsymbol{\nu}(x)\|}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|}dx}{b/2}=\frac{1}{\boldsymbol{\pi}|\boldsymbol{r}|}.\tag{5.4}$$

Similarly, we have

$$\lim_{b\to\infty}\frac{\int_0^{b/2}\frac{\|\tilde{\boldsymbol{\nu}}(x)\|}{\|\tilde{\boldsymbol{\nu}}(b/2)\Delta_{|\boldsymbol{r}|}\|}dx}{b/2}=\frac{1}{\boldsymbol{\pi}|\boldsymbol{r}|}.\tag{5.5}$$

From Lemma 5.3, (5.4) and (5.5), we have

$$\frac{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}{c_b}\sim\frac{\boldsymbol{\pi}|\boldsymbol{r}|}{b},\tag{5.6}$$

where $f(b)\sim g(b)$ means that $\lim_{b\to\infty}f(b)/g(b)=1$. By Corollary 5.2, for sufficiently large b, we have

$$\frac{\boldsymbol{\nu}^{(b)}(b-x)}{c_b}\sim\frac{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}{c_b}\frac{1}{\|\tilde{\boldsymbol{\nu}}(b/2)\Delta_{|\boldsymbol{r}|}\|}\tilde{\boldsymbol{\nu}}(x),\quad x\in(0,b/2),\tag{5.7}$$

From Lemma 4.3 and (5.6), (5.7) reduces to

$$\frac{\boldsymbol{\nu}^{(b)}(b-x)}{c_b} \sim \frac{1}{b}\frac{1}{\overline{\boldsymbol{s}C}^{+-}\Delta_{\boldsymbol{\pi}^-}^{-1}(-\Delta_{\boldsymbol{r}^-})^{-1}(\tilde{\boldsymbol{\beta}}^-)^{\mathrm{T}}}\tilde{\boldsymbol{\nu}}(x), \quad x \in (0, b/2),$$

where $\tilde{\boldsymbol{\beta}}^-$ is the stationary distribution of $\tilde{Q}^{(-)}$. Thus, we have

$$\frac{\boldsymbol{\nu}^{(b)+}(b-)}{c_b} \sim \frac{1}{b}\frac{1}{\overline{\boldsymbol{s}C}^{+-}\Delta_{\boldsymbol{\pi}^-}^{-1}(-\Delta_{\boldsymbol{r}^-})^{-1}(\tilde{\boldsymbol{\beta}}^-)^{\mathrm{T}}}\tilde{\boldsymbol{\nu}}^+(0+),$$

which completes the proof of Theorem 5.2 by (3.4) and (3.7).

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Appendix

A Proof of Lemma 2.2

For $x > 0$, let $N_0^{+\bullet}(0,x)$ be the $|\mathcal{S}^+| \times |\mathcal{S}|$ matrix whose (i,j)-th element is given by

$$\left[N_0^{+\bullet}(0,x)\right]_{ij} = E\left[\sum_{0\le t<\tau_0} 1(X(t)=x, M(t)=j)\middle| X(0)=0, M(0)=i\right], \tag{A.1}$$

which is the expected number of level crossing at level x with background state $j \in \mathcal{S}$ before returning to level 0, starting from level 0 with background state $i \in \mathcal{S}^+$. In what follows, we show that

$$\varphi_0^{+\bullet}(0,x)\Delta_{|\boldsymbol{r}|} = N_0^{+\bullet}(0,x). \tag{A.2}$$

Since $\varphi_0^{+\bullet}(0,x)$ is the derivative of $\Phi_0^{+\bullet}(0,x)$ by Corollary 2.2, we have

$$\begin{aligned}
[\varphi_0^{+\bullet}(0,x)]_{ij} &= \liminf_{h\to 0} \frac{1}{h} E\left[\int_0^{\tau_0} 1(X(t)\in(x,x+h], M(t)=j)dt \middle| X(0)=0, M(0)=i\right] \\
&\ge E\left[\liminf_{h\to 0} \frac{1}{h}\int_0^{\tau_0} 1(X(t)\in(x,x+h], M(t)=j)dt \middle| X(0)=0, M(0)=i\right] \\
&= \frac{1}{r_j} E\left[\liminf_{h\to 0} \frac{1}{h}\int_0^{\tau_0} 1(X(t)\in(x,x+h], M(t)=j)dX(t) \middle| X(0)=0, M(0)=i\right] \\
&= \frac{1}{r_j} E\left[\sum_{0\le t<\tau_0} 1(X(t)=x, M(t)=j)\middle| X(0)=0, M(0)=i\right] \\
&= \frac{1}{r_j}[N_0^{+\bullet}(0,x)]_{ij}
\end{aligned}$$

for $i,j \in \mathcal{S}^+$, where the first inequality follows by Fatou's lemma, and the second equality follows since $r_{M(t)}dt = dX(t)$ except for time epoch that $X(t)$ changes. When $j \in \mathcal{S}^-$, we can get the similar formula. Then, we have

$$\varphi_0^{+\bullet}(0,x) \ge N_0^{+\bullet}(0,x)\Delta_{|\boldsymbol{r}|}^{-1}. \tag{A.3}$$

We next prove the reverse direction of the inequality in (A.3). To this end, we use the truncation argument as follows. For $a, x > 0$, $i \in \mathcal{S}^+$ and $j \in \mathcal{S}$, let $\varphi_0^{(a)+\bullet}(0,x)$ be the $|\mathcal{S}^+| \times |\mathcal{S}|$ matrix whose (i,j)-th element is given by

$$[\varphi_0^{(a)+\bullet}(0,x)]_{ij} = \liminf_{h\to 0} \frac{1}{h} E\left[\int_0^{\tau_0 \wedge a} 1(X(t) \in (x, x+h], M(t) = j) dt \middle| X(0) = 0, M(0) = i \right]$$

The interchange of the limit and the expectation is guaranteed as follows. Assume that $j \in \mathcal{S}^+$. Let $C_{(x,j)}$ be the number of crossing time at level x with background state j before time $\tau_0 \wedge a$. Let $D_{(x,j)}$ be the number of state transitions that the background state changes to state j while the level process stays in $(x, x+h]$ before time $\tau_0 \wedge a$. Then we have

$$\frac{1}{h} \int_0^{\tau_0 \wedge a} 1(X(t) \in (x, x+h], M(t) = j) dt \le \frac{1}{r_j} (C_{(x,j)} + D_{(x,j)}).$$

Since $E[C_{(x,j)} + D_{(x,j)} | X(0) = 0, M(0) = i]$ is finite, we have

$$[\varphi_0^{(a)+\bullet}(0,x)]_{ij} = E\left[\liminf_{h\to 0} \frac{1}{h} \int_0^{\tau_0 \wedge a} 1(X(t) \in (x, x+h], M(t) = j) dt \middle| X(0) = 0, M(0) = i \right]$$

by the dominated convergence theorem. When $j \in \mathcal{S}^-$, we can get the similar formula. Hence, we have

$$\varphi_0^{(a)+\bullet}(0,x) \le N_0^{+\bullet}(0,x) \Delta_{|\boldsymbol{r}|}^{-1}. \tag{A.4}$$

Finally, we show that

$$\lim_{a\to\infty} \varphi_0^{(a)+\bullet}(0,x) = \varphi_0^{+\bullet}(0,x), \tag{A.5}$$

which completes the proof of the lemma by (A.3) and (A.4). For $a, x > 0$, let $\Phi_0^{(a)+\bullet}(0,x)$ be the $|\mathcal{S}^+| \times |\mathcal{S}|$ matrix whose (i,j)-th element is given by

$$[\Phi_0^{(a)+\bullet}(0,x)]_{ij} = E\left[\int_0^{\tau_0 \wedge a} 1(X(t) \le x, M(t) = j) dt \middle| X(0) = 0, M(0) = i \right].$$

Similarly to the proof of Lemma 2.1, we have

$$\pi_i [\Phi_0^{(a)+\bullet}(0,x)]_{ij} = \pi_j \frac{1}{r_i} \int_0^x P(\hat{M}(\hat{\tau}_y \wedge a) = i | \hat{M}(0) = j) dy.$$

By differentiating the above formula, we have

$$\pi_i [\varphi_0^{(a)+\bullet}(0,x)]_{ij} = \pi_j \frac{1}{r_i} P(\hat{M}(\hat{\tau}_x \wedge a) = i | \hat{M}(0) = j).$$

The monotone convergence theorem implies that

$$\begin{aligned} \lim_{a\to\infty} [\varphi_0^{(a)+\bullet}(0,x)]_{ij} &= \frac{\pi_j}{\pi_i} \frac{1}{r_i} P(\hat{M}(\hat{\tau}_x) = i | \hat{M}(0) = j) \\ &= \frac{\pi_j}{\pi_i} \frac{1}{r_i} [\hat{\Psi}_{-\infty}^{\bullet+}(0,x)]_{ji}. \end{aligned}$$

Then, we obtain (A.5) by Corollary 2.1 and (2.4).

B Proof of Lemma 2.3

From the definition of $N_0^{++}(0,x)$ (see, (A.1) in Appendix A), we have

$$N_0^{++}(0,x) = \Psi_0^{++}(0,x)(I - \Psi^{+-}\Psi_{-x}^{-+}(0,0))^{-1}. \tag{B.1}$$

By conditioning on the first time that the Markov additive process returns to initial level, we have

$$\Psi_{-\infty}^{++}(0,x) = \Psi_0^{++}(0,x) + \Psi_x^{+-}(0,0)\Psi^{-+}\Psi_{-\infty}^{++}(0,x). \tag{B.2}$$

From (B.1), (B.2) and (2.1), we have

$$N_0^{++}(0,x)(I - \Psi^{+-}\Psi_{-x}^{-+}(0,0)) = (I - \Psi_x^{+-}(0,0)\Psi^{-+})\exp(xQ^{(+)}).$$

Since we have assumed that $d = 0$, $Q^{(+)}$ is a non-defective matrix, and Ψ^{+-} and Ψ^{-+} are stochastic matrices. Denote the stationary distribution of $Q^{(+)}$ by $\boldsymbol{\kappa}^+$, then we have

$$\begin{aligned}(\lim_{x\to\infty} N_0^{++}(0,x))(I - \Psi^{+-}\Psi^{-+}) &= (I - \Psi^{+-}\Psi^{-+})\mathbf{1}\boldsymbol{\kappa}^+ \\ &= O,\end{aligned}$$

which implies that

$$\lim_{x\to\infty} N_0^{++}(0,x) = \lim_{x\to\infty} N_0^{++}(0,x)\Psi^{+-}\Psi^{-+}. \tag{B.3}$$

From (2.4) and (A.2), and noting that $\hat{Q}^{(+)}$ is non-defective matrix for $d = 0$, (B.3) reduces to

$$(\hat{\boldsymbol{\kappa}}^+)^{\mathrm{T}}(\boldsymbol{r}^+)^{\mathrm{T}}\Delta_{\boldsymbol{\pi}^+} = (\hat{\boldsymbol{\kappa}}^+)^{\mathrm{T}}(\boldsymbol{r}^+)^{\mathrm{T}}\Delta_{\boldsymbol{\pi}^+}\Psi^{+-}\Psi^{-+}, \tag{B.4}$$

where $\hat{\boldsymbol{\kappa}}^+$ is the stationary distribution of $\hat{Q}^{(+)}$. Noting that $\boldsymbol{\xi}^+$ is the stationary distribution of $\Psi^{+-}\Psi^{-+}$, we have $\boldsymbol{\xi}^+ = m_1(\boldsymbol{r}^+)^{\mathrm{T}}\Delta_{\boldsymbol{\pi}^+}$ by (B.4), where m_1 is a normalizing constant. Similarly, we have $\boldsymbol{\xi}^- = m_2(-\boldsymbol{r}^-)^{\mathrm{T}}\Delta_{\boldsymbol{\pi}^-}$ for a normalizing constant m_2. Because of $d = 0$, we have $\boldsymbol{\xi}^+\mathbf{1} = \boldsymbol{\xi}^-\mathbf{1} = 1/2$, which implies that

$$m_1 = m_2 = \frac{1}{\boldsymbol{\pi}^+\boldsymbol{r}^+ - \boldsymbol{\pi}^-\boldsymbol{r}^-}.$$

C Proof of Lemma 3.2

Similarly to Lemma 2.1, for $i \in \mathcal{S}^-$, $j \in \mathcal{S}$ and $x \in (0,b)$, we have

$$\begin{aligned}[\Phi_{0b}^{-\bullet}(b,x)]_{ij} &= E\left[\int_0^{\tau_{X(0)}\wedge\tau_{X(0)-b}} 1(X(t) \le X(0) - (b-x), M(t) = j)dt \middle| M(0) = i\right] \\ &= \frac{1}{\pi_i}\int_0^\infty P\begin{pmatrix} -b < \hat{X}(t) - \hat{X}(0) \le -(b-x), \hat{M}(0) = j, \hat{M}(t) = i, \\ \max_{u\in(0,t)}\hat{X}(u) - b < \hat{X}(t) < \min_{u\in(0,t)}\hat{X}(u)\end{pmatrix} dt \\ &= \frac{\pi_j}{|r_i|\pi_i}\int_{-b}^{-(b-x)} P(\hat{M}(\hat{\tau}_y) = i, \hat{\tau}_y \le \hat{\eta}^{(b)}|\hat{M}(0) = j)dy, \end{aligned} \tag{C.1}$$

where $\hat{\eta}^{(b)} \equiv \min\{t > 0 | \max_{u\in(0,t)} \hat{X}(u) - \min_{u\in(0,t)} \hat{X}(u) = b\}$. The above equation implies that $\Phi_{0b}^{-\bullet}(b,x)$ is differentiable with respective to $x \in (0,b)$. Similarly, for $i \in \mathcal{S}^+, j \in \mathcal{S}$ and $x \in (0,b)$, we have

$$[\Phi_{0b}^{+\bullet}(0,x)]_{ij} = \frac{\pi_j}{r_i \pi_i} \int_0^x P(\hat{M}(\hat{\tau}_y) = i, \hat{\tau}_y \le \hat{\eta}^{(b)} | \hat{M}(0) = j) dy, \tag{C.2}$$

which implies that $\Phi_{0b}^{+\bullet}(0,x)$ is differentiable.

D Expression of $\boldsymbol{\nu}^{(b)}(x)$ for $d \neq 0$

By conditioning on the first time that the level process hits the level b for $x \in (0,b)$, we have

$$\begin{aligned} \Phi_0^{+\bullet}(0,x) &= \Phi_{0b}^{+\bullet}(0,x) + \Psi_0^{++}(0,b)\Psi^{+-}\Phi_0^{-\bullet}(b,x), \\ \Phi_0^{-\bullet}(b,x) &= \Phi_{0b}^{-\bullet}(b,x) + \Psi_0^{-+}(b,b)\Psi^{+-}\Phi_0^{-\bullet}(b,x), \end{aligned}$$

which reduce to

$$\Phi_0^{+\bullet}(0,x) = \Phi_{0b}^{+\bullet}(0,x) + \Psi_0^{++}(0,b)\Psi^{+-}(I - \Psi_0^{-+}(b,b)\Psi^{+-})^{-1}\Phi_{0b}^{-\bullet}(b,x). \tag{D.1}$$

Similarly, for $x \in (0,b)$, we have

$$\Phi_b^{-\bullet}(b,x) = \Phi_{0b}^{-\bullet}(b,x) + \Psi_b^{--}(b,0)\Psi^{-+}(I - \Psi_b^{+-}(0,0)\Psi^{-+})^{-1}\Phi_{0b}^{+\bullet}(0,x). \tag{D.2}$$

From Lemma 2.2, we have

$$\varphi_0^{+-}(0,b)\Delta_{|\boldsymbol{r}^-|} = \Psi_0^{++}(0,b)\Psi^{+-}(I - \Psi_0^{-+}(b,b)\Psi^{+-})^{-1}. \tag{D.3}$$

Let $\tilde{\varphi}_0^{-+}(0,x)$ be the density of the occupation measure for the level reversed additive process $(-X(t), M(t))$. Similarly to (D.3), we have

$$\tilde{\varphi}_0^{-+}(0,b)\Delta_{\boldsymbol{r}^+} = \Psi_b^{--}(b,0)\Psi^{-+}(I - \Psi_b^{+-}(0,0)\Psi^{-+})^{-1}. \tag{D.4}$$

From (D.1), (D.2), (D.3) and (D.4), we have

$$\begin{pmatrix} \Phi_{0b}^{-\bullet}(b,x) \\ \Phi_{0b}^{+\bullet}(0,x) \end{pmatrix} = \begin{pmatrix} I & \tilde{\varphi}_0^{-+}(0,b)\Delta_{\boldsymbol{r}^+} \\ \varphi_0^{+-}(0,b)\Delta_{|\boldsymbol{r}^-|} & I \end{pmatrix}^{-1} \begin{pmatrix} \Phi_b^{-\bullet}(b,x) \\ \Phi_0^{+\bullet}(0,x) \end{pmatrix}$$

where the nonsingularity of the matrix is ensured by $d \neq 0$ (see Lemma 4.2 of [5]). By differentiating the above formula, we have

$$\begin{pmatrix} \varphi_{0b}^{-\bullet}(b,x) \\ \varphi_{0b}^{+\bullet}(0,x) \end{pmatrix} = \begin{pmatrix} I & \tilde{\varphi}_0^{-+}(0,b)\Delta_{\boldsymbol{r}^+} \\ \varphi_0^{+-}(0,b)\Delta_{|\boldsymbol{r}^-|} & I \end{pmatrix}^{-1} \begin{pmatrix} \varphi_b^{-\bullet}(b,x) \\ \varphi_0^{+\bullet}(0,x) \end{pmatrix}. \tag{D.5}$$

From (D.5) and $\varphi_b^{-\bullet}(b,x) = \tilde{\varphi}_0^{-\bullet}(0,b-x)$, the right hand side of (3.5) can be rewritten by

$$\begin{pmatrix} \underline{\boldsymbol{s}}^{(b)} & \overline{\boldsymbol{s}}^{(b)} \end{pmatrix} \begin{pmatrix} \underline{C}^{-+} & O \\ O & \overline{C}^{+-} \end{pmatrix} \begin{pmatrix} I & \tilde{\varphi}_0^{-+}(0,b)\Delta_{\boldsymbol{r}^+} \\ \varphi_0^{+-}(0,b)\Delta_{|\boldsymbol{r}^-|} & I \end{pmatrix}^{-1} \begin{pmatrix} \tilde{\varphi}_0^{-\bullet}(0,b-x) \\ \varphi_0^{+\bullet}(0,x) \end{pmatrix}.$$

Since $\varphi_0^{+\bullet}(0,x)$ and $\tilde{\varphi}_0^{-\bullet}(0,x)$ have the matrix exponential form (see Corollary 2.2), the above formula is equivalent to the ones obtained in Theorem 4.4 of [5] when $\boldsymbol{r}^+ = \boldsymbol{1}$ and $\boldsymbol{r}^- = -\boldsymbol{1}$.

E Proof of Lemma 3.3

By (3.3), (3.7) is rewritten by

$$\underline{\boldsymbol{s}}^{(b)}\underline{C}^{-+}\varphi_{0b}^{++}(0,b-)\Delta_{\boldsymbol{r}^+} + \overline{\boldsymbol{s}}^{(b)}(\overline{C}^{++} + \overline{C}^{+-}\varphi_{0b}^{-+}(b,b-)\Delta_{\boldsymbol{r}^+}) = \mathbf{0}. \tag{E.1}$$

We show that (E.1) is equivalent to (3.2) through the following two lemmas.

Lemma E.1

$$\varphi_{0b}^{++}(0,b-) = \Delta_{\boldsymbol{r}^+}^{-1}\Delta_{\boldsymbol{\pi}^+}^{-1}\hat{\Psi}_0^{++}(0,b)^{\mathrm{T}}\Delta_{\boldsymbol{\pi}^+}, \tag{E.2}$$

$$\varphi_{0b}^{-+}(b,b-) = \Delta_{|\boldsymbol{r}^-|}^{-1}\Delta_{\boldsymbol{\pi}^-}^{-1}\hat{\Psi}_b^{+-}(0,0)^{\mathrm{T}}\Delta_{\boldsymbol{\pi}^+}. \tag{E.3}$$

Proof. By differentiating (C.1), we have

$$\pi_i[\varphi_{0b}^{-\bullet}(b,x)]_{ij} = \frac{\pi_j}{|r_i|}P(\hat{M}(\hat{\tau}_{-(b-x)}) = i, \hat{\tau}_{-(b-x)} \le \hat{\eta}^{(b)}|\hat{M}(0) = j).$$

Then, we have

$$\begin{aligned}\pi_i[\varphi_{0b}^{-+}(b,b-)]_{ij} &= \frac{\pi_j}{|r_i|}P(\hat{M}(\hat{\tau}_0) = i, \hat{\tau}_0 \le \hat{\eta}^{(b)}|\hat{M}(0) = j)\\ &= \frac{\pi_j}{|r_i|}[\hat{\Psi}_b^{+-}(0,0)]_{ji}\end{aligned}$$

for $i \in \mathcal{S}^-$ and $j \in \mathcal{S}^+$, which implies (E.3). Similarly, (E.2) can be derived from (C.2). □

It is not hard to see that the following result is obtained by using the same techniques used in Corollary 2.2 of [9]

Lemma E.2

$$\Psi_0^{++}(0,b) = \Delta_{\boldsymbol{r}^+}^{-1}\Delta_{\boldsymbol{\pi}^+}^{-1}\hat{\Psi}_0^{++}(0,b)^{\mathrm{T}}\Delta_{\boldsymbol{\pi}^+}\Delta_{\boldsymbol{r}^+},$$

$$\Psi_0^{-+}(b,b) = \Delta_{|\boldsymbol{r}^-|}^{-1}\Delta_{\boldsymbol{\pi}^-}^{-1}\hat{\Psi}_b^{+-}(0,0)^{\mathrm{T}}\Delta_{\boldsymbol{\pi}^+}\Delta_{\boldsymbol{r}^+}.$$

Hence, (E.1) is equivalent to (3.2) by Lemma E.1 and Lemma E.2.

F Proof of Theorem 5.1

We first prove (i). Assuming that $d < 0$, we have $\lim_{b\to\infty} c_b < \infty$. From (3.1), (3.2) and (3.4), the probability vectors $\underline{\boldsymbol{p}}^{(b)}$ and $\overline{\boldsymbol{p}}^{(b)}$ satisfy

$$\underline{\boldsymbol{p}}^{(b)}(\underline{C}^{--} + \underline{C}^{-+}\Psi_b^{+-}(0,0)) + \overline{\boldsymbol{p}}^{(b)}\overline{C}^{+-}\Psi_b^{--}(b,0) = \mathbf{0}, \tag{F.1}$$

$$\underline{\boldsymbol{p}}^{(b)}\underline{C}^{-+}\Psi_0^{++}(0,b) + \overline{\boldsymbol{p}}^{(b)}(\overline{C}^{++} + \overline{C}^{+-}\Psi_0^{-+}(b,b)) = \mathbf{0}. \tag{F.2}$$

We next show that $\overline{\boldsymbol{p}}^{(b)}$ converges to zero as b gets large. By conditioning on the first time that the level process hits level b, the hitting probability $\Psi_0^{++}(0,b)$ satisfies

$$\Psi_{-\infty}^{++}(0,b) = \Psi_0^{++}(0,b) + \Psi_b^{+-}(0,0)\Psi^{-+}\Psi_{-\infty}^{++}(0,b).$$

Then, we have

$$\Psi_0^{++}(0,b) = (I - \Psi_b^{+-}(0,0)\Psi^{-+})\exp(bQ^{(+)}) \tag{F.3}$$

from Proposition 2.1. Since $Q^{(+)}$ is defective for $d < 0$, we have

$$\lim_{b\to\infty} \Psi_0^{++}(0,b) = 0.$$

Hence, from (F.2) and (F.3), we have

$$\lim_{b\to\infty} \overline{\boldsymbol{p}}^{(b)}(\overline{C}^{++} + \overline{C}^{+-}\Psi^{-+}) = \mathbf{0},$$

which implies that

$$\lim_{b\to\infty} \overline{\boldsymbol{p}}^{(b)} = \mathbf{0} \tag{F.4}$$

since Ψ^{-+} is substochastic for $d < 0$, that is, $\overline{C}^{++} + \overline{C}^{+-}\Psi^{-+}$ is a defective matrix. From (F.1) and (F.4), $\underline{\boldsymbol{p}}^{(b)}$ converges to $\underline{\boldsymbol{p}}$, which is the stationary measure of $\underline{C}^{--} + \underline{C}^{-+}\Psi^{+-}$ satisfying the normalizing condition

$$\underline{\boldsymbol{p}}\mathbf{1} + \int_0^\infty \boldsymbol{\nu}(x)\mathbf{1} = 1,$$

that is,

$$\underline{\boldsymbol{p}}\mathbf{1} + \underline{\boldsymbol{p}}\underline{C}^{-+}\Delta_{\boldsymbol{\pi}^+}^{-1}\Delta_{\boldsymbol{r}^+}^{-1}\begin{pmatrix} \hat{\Psi}^{-+}(-\hat{Q}^{(+)})^{-1} \\ (-\hat{Q}^{(+)})^{-1} \end{pmatrix}^{\mathrm{T}} \Delta_{\boldsymbol{\pi}}\mathbf{1} = 1$$

by Lemma 4.1, where $(-\hat{Q}^{(+)})^{-1}$ exists since $\hat{Q}^{(+)}$ is defective for $d < 0$. We finally show that $\overline{\boldsymbol{p}}^{(b)}$ exponentially converges to zero. From (F.2), we have

$$\overline{\boldsymbol{p}}^{(b)} = \underline{\boldsymbol{p}}^{(b)}\underline{C}^{-+}\Psi_0^{++}(0,b)(-\overline{C}^{++} - \overline{C}^{+-}\Psi_0^{-+}(b,b))^{-1}. \tag{F.5}$$

As $b \to \infty$, $\Psi_0^{-+}(b,b)$ converges to Ψ^{-+}, which is substochastic for $d < 0$. We complete the proof by showing that $\Psi_0^{++}(0,b)$ exponentially converges to zero. Let $-\alpha < 0$ be the Perron Frobeneus eigenvalue of $Q^{(+)}$ with a corresponding right eigenvector $\boldsymbol{g}^+ > \mathbf{0}$, i.e., $Q^{(+)}\boldsymbol{g}^+ = -\alpha\boldsymbol{g}^+$. Since $\Delta_{\boldsymbol{g}^+}^{-1}(\alpha I + Q^{(+)})\Delta_{\boldsymbol{g}^+}$ is a non-defective transition rate matrix of a Markov chain, we have

$$\lim_{b\to\infty} \exp(b\Delta_{\boldsymbol{g}^+}^{-1}(\alpha I + Q^{(+)})\Delta_{\boldsymbol{g}^+}) = \mathbf{1}^+\boldsymbol{\eta}^+, \tag{F.6}$$

where $\boldsymbol{\eta}^+$ is the stationary distribution of $\Delta_{\boldsymbol{g}^+}^{-1}(\alpha I + Q^{(+)})\Delta_{\boldsymbol{g}^+}$. Combining (F.3), (F.5) and (F.6) completes the proof.

We next assume that $d > 0$. Similarly to the preceding argument, we have $\lim_{b\to\infty} \underline{\boldsymbol{p}}^{(b)} = \mathbf{0}$ and $\lim_{b\to\infty} \overline{\boldsymbol{p}}^{(b)} = \overline{\boldsymbol{p}}$, where $\overline{\boldsymbol{p}}$ is the stationary measure of the non-defective transition rate matrix $\overline{C}^{++} + \overline{C}^{+-}\Psi^{-+}$ satisfying the normalizing condition

$$\overline{\boldsymbol{p}}\mathbf{1} + \overline{\boldsymbol{p}}\overline{C}^{+-}\Delta_{\boldsymbol{\pi}^-}^{-1}\Delta_{|\boldsymbol{r}^-|}^{-1}\begin{pmatrix} (-\tilde{Q}^{(-)})^{-1} \\ \tilde{\Psi}^{+-}(-\tilde{Q}^{(-)})^{-1} \end{pmatrix}^{\mathrm{T}} \Delta_{\boldsymbol{\pi}}\mathbf{1} = 1.$$

G Proof of lemmas in Section 5

(Proof of Lemma 5.1) Similarly to (5.2), we have

$$\frac{[\boldsymbol{\xi}]_j}{[\boldsymbol{\xi}]_i} = E\left[\sum_{0\le t<\tau_0(i)} 1\left((X(t),M(t))=(0,j)\right)\middle|(X(0),M(0))=(0,i)\right] \tag{G.1}$$

for $i,j\in\mathcal{S}$. From (5.2) and the transition structure (i) of the finite fluid queue, we have

$$\frac{[\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}]_j}{[\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}]_i} \ge E\left[\sum_{0\le t<\tau_0(i)\wedge\tau_{-b/2,b/2}} 1\left((X(t),M(t))=(0,j)\right)\middle|(X(0),M(0))=(0,i)\right]. \tag{G.2}$$

From (G.2) and (G.1), we have

$$\liminf_{b\to\infty}\frac{[\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}]_j}{[\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}]_i} \ge \frac{[\boldsymbol{\xi}]_j}{[\boldsymbol{\xi}]_i}.$$

By interchanging the roles of i and j, we obtain another direction of the inequality, which completes the proof.

(Proof of Lemma 5.2) We partition $\boldsymbol{\nu}^{(b)}(x)$ and $\boldsymbol{\nu}(x)$ according to $\mathcal{S}^-$ and $\mathcal{S}^+$ as

$$\boldsymbol{\nu}^{(b)}(x)=(\boldsymbol{\nu}^{(b)-}(x),\boldsymbol{\nu}^{(b)+}(x)),\quad \boldsymbol{\nu}(x)=(\boldsymbol{\nu}^-(x),\boldsymbol{\nu}^+(x)).$$

From (5.1), for $x\in(0,b/2)$, we have

$$\boldsymbol{\nu}^{(b)}(x)\Delta_{|\boldsymbol{r}|}=\boldsymbol{\nu}^{(b)-}(b/2)\Delta_{|\boldsymbol{r}^-|}M^{-\bullet}_{b/2}(b/2,x),\quad \boldsymbol{\nu}(x)\Delta_{|\boldsymbol{r}|}=\boldsymbol{\nu}^-(b/2)\Delta_{|\boldsymbol{r}^-|}M^{-\bullet}_{b/2}(b/2,x),$$

where $M^{-\bullet}_{b/2}(b/2,x)$ is the $|\mathcal{S}^-|\times|\mathcal{S}|$ matrix whose (i,j)-th element is given by

$$[M^{-\bullet}_{b/2}(b/2,x)]_{ij}=E\left[\sum_{0\le t<\sigma_{b/2}}1(Y(t)=x,J(t)=j)\middle|Y(0)=b/2,J(0)=i\right],\quad i\in\mathcal{S}^-,j\in\mathcal{S},$$

where $\sigma_{b/2}=\inf\{t>0|Y(t)=b/2\}$. Then, we have

$$\frac{\boldsymbol{\nu}^{(b)}(x)\Delta_{|\boldsymbol{r}|}}{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|} = \frac{\boldsymbol{\nu}^{(b)-}(b/2)\Delta_{|\boldsymbol{r}^-|}}{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}M^{-\bullet}_{b/2}(b/2,x), \tag{G.3}$$

$$\frac{\boldsymbol{\nu}(x)\Delta_{|\boldsymbol{r}|}}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|} = \frac{\boldsymbol{\nu}^{-}(b/2)\Delta_{|\boldsymbol{r}^-|}}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|}M^{-\bullet}_{b/2}(b/2,x). \tag{G.4}$$

By applying Lemma 4.2 and Lemma 5.1 to (G.3) and (G.4), we complete the proof.

(Proof of Lemma 5.3) By Lemma 5.2 and Corollary 5.2, for arbitrary $\epsilon>0$, there exists $\delta>0$ such that

$$\frac{1}{1+\epsilon}\int_0^{b/2}\|\boldsymbol{\nu}^{(b)}(x)\|dx \le \frac{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|}\int_0^{b/2}\|\boldsymbol{\nu}(x)\|dx \le (1+\epsilon)\int_0^{b/2}\|\boldsymbol{\nu}^{(b)}(x)\|dx,$$

$$\frac{1}{1+\epsilon}\int_{b/2}^{b}\|\boldsymbol{\nu}^{(b)}(x)\|dx \le \frac{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}{\|\tilde{\boldsymbol{\nu}}(b/2)\Delta_{|\boldsymbol{r}|}\|}\int_0^{b/2}\|\tilde{\boldsymbol{\nu}}(x)\|dx \le (1+\epsilon)\int_{b/2}^{b}\|\boldsymbol{\nu}^{(b)}(x)\|dx$$

for $b \geq \delta$. By combining the above inequalities and Lemma 3.1, we have

$$\begin{aligned}
&\frac{1}{1+\epsilon}(c_b - \|\underline{\boldsymbol{s}}^{(b)}\| - \|\overline{\boldsymbol{s}}^{(b)}\|) \\
&\leq \frac{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}{\|\boldsymbol{\nu}(b/2)\Delta_{|\boldsymbol{r}|}\|} \int_0^{b/2} \|\boldsymbol{\nu}(x)\|dx + \frac{\|\boldsymbol{\nu}^{(b)}(b/2)\Delta_{|\boldsymbol{r}|}\|}{\|\tilde{\boldsymbol{\nu}}(b/2)\Delta_{|\boldsymbol{r}|}\|} \int_0^{b/2} \|\tilde{\boldsymbol{\nu}}(x)\|dx \\
&\leq (1+\epsilon)(c_b - \|\underline{\boldsymbol{s}}^{(b)}\| - \|\overline{\boldsymbol{s}}^{(b)}\|)
\end{aligned}$$

for $b \geq \delta$. Dividing the above inequalities by c_b and taking limit with respect to b complete the proof of the lemma since c_b gets large for $d = 0$.