Exact asymptotics for a Lévy-driven tandem queue with an intermediate input

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Abstract

We consider a Lévy-driven tandem queue with an intermediate input assuming that its buffer content process obtained by a reflection mapping has the stationary distribution. For this queue, no closed form formula is known, not only for its distribution but also for the corresponding transform. In this paper we consider only light-tailed inputs, and derive exact asymptotics for the tail distribution of convex type combination of two buffer contents. This includes the marginal stationary distributions of the buffer contents and their sum as special cases. These results generalize those of Lieshout and Mandjes from the recent paper [13] for the corresponding tandem queue without an intermediate input.

1 Introduction

Brownian fluid networks have been studied for many years, and they are well understood as reflected Brownian motions (see, e..g., [5, 8, 9]). They can be easily extended to spectrally positive Lévy inputs. These extended networks are referred here to as Lévy driven network queues. In applications of those networks, it may be interesting to look for their stationary behavior, provided the existence of their stationary distributions are assumed. Since it is hard to get the stationary distributions except for special cases, asymptotic tail behavior of those distributions is interesting. However, it is still hard to get the asymptotics such as tail decay rates from primitive modeling data even for two node fluid networks although there are some notable results for rough asymptotics on the two dimensional reflected Brownian motions (see, e.g., [2]). This is contrasted with their discrete time version, so called a reflected random walk in a nonnegative two dimensional quadrant, for which the asymptotic behavior has been studied (see, e.g., [4, 17]).

Recently, Lieshout and Mandjes [12, 13] solve this asymptotic decay problem for a Lévy-driven two node tandem queue when there is no intermediate input, which means that the second queue has no exogenous input. For this tandem queue, the joint Laplace transform of the stationary distribution is obtained in closed form by Dębicki, Dieker and Rolski [6]. Using this closed form expression, Lieshout and Mandjes [13] get rough decay rates in all directions and exact asymptotics for distribution of separate nodes or their convex combinations. Here, the tail probability is said to have rough decay rate α if its logarithm at level x divided by x converges to $-\alpha$ as $x \to \infty$, and said to have exact asymptotics h for some function h if the ratio of the tail probability at level x and $h(x)$ converges to one.

The results in [13] are very interesting since the rough decay rates and exact asymptotics h can be computed. However, their approach heavily depends on the explicit form of the Laplace transform, which is only available for a pure tandem case. For example, if an intermediate input is added, we cannot get such a closed form formula for the Laplace transform. We consider this tandem queue. This model is more flexible for applications. Its asymptotic behavior is theoretically interesting since nothing is known about the stationary distributions except for some basic facts such as stability.

We solve this asymptotic decay problem mainly for the marginal stationary distributions. Since the marginal distribution for the first node is obtained in term of Laplace transform as the Pollaczek-Khinchine formula, our main interest is in exact asymptotics on the second node. These results exhibit some analogy to the discrete time counterpart, that is, the reflected random walk. Similar exact asymptotics are also reported for hitting probabilities of two dimensional risk processes in [3]. As a by product, we also get exact asymptotics when a convex combination of the two buffer contents gets large.

The approach of this paper is analytic, and exact asymptotics is studied only for marginal distributions. This enables us to use classical results on one dimensional distributions through their transforms. In other words, our results may not be so informative on sample path behavior, which has been extensively studied in the large deviations theory. Nevertheless, the results has multidimensional, precisely, two-dimensional, feature. For example, we identify the domain of the moment generation function for the two-dimensional stationary distribution. This may be related to the rate function in the sample path large deviations. We hope the present results would be also useful for the large deviations theory.

This paper is composed of six sections. In Section 2, we first derive stationary equation using Itô's integral formula, and show that the recent result from [6] is easily obtained from the stationary equation. In Sections 3 and 4, we assume that there is no jump input at both nodes for simplicity. In Section 3, we identify the domain of the moment generating function of the joint buffer contents. In Section 4, exact asymptotics are obtained. In Section 5, we outline how those results can be extended when there are jump inputs at both nodes. We finally discuss consistency with existing results and possible extensions of the presented results in Section 6.

2 Stationary equation for moment generating functions (MGFs)

We now formally introduce a Lévy-driven tandem fluid queue with an intermediate input. This tandem queue has two nodes, numbered as 1 and 2. Both nodes has exogenous input processes, and constant processing rates. Outflow from node 1 goes to node 2, and outflow from node 2 leaves the system. As usual, we always assume that all processes are right-continuous and have left-hand limits.

We assume that those exogenous inputs are independent Lévy processes of the form: for node i

$$X_i(t) = a_i t + B_i(t) + J_i^{(0)}(t) + J_i^{(1)}(t), \qquad i = 1, 2, \tag{2.1}$$

where a_i is a nonnegative constant, $B_i(t)$ is a Brownian motion with variance σ_i^2 and null drift, and $J_i^{(0)}(t)$ and $J^{(1)}$ are given by

$$J_i^{(0)}(t) = \int_0^t \int_0^1 x(\Lambda_i(du, dx) - du\nu_i(dx)),$$
$$J_i^{(1)}(t) = \int_0^t \int_{1+}^\infty x\Lambda_i(du, dx),$$

where Λ_i is a time homogeneous compound Poisson process and $\nu_i(dx) = E(\Lambda_i((0, 1], dx))$ such that

$$\int_0^\infty \min(x^2, 1)\nu_i(dx) < \infty, \qquad i = 1, 2,$$

and Λ_1 and Λ_2 are independent of each other as well as of the other processes. It should be noted that $J_i^{(0)}(t)$ is a martingale with respect to filtration $\mathcal{F}_t^{\boldsymbol{X}}$ generated by $\{\boldsymbol{X}(t); t \geq 0\} \equiv \{(X_1(t), X_2(t)); t \geq 0\}$. It is known that the one-dimensional Levy process has the form (2.1) (e.g., see [11, 18]).

Denote the Lévy exponent of $X_i(t)$ by $\kappa_i(\cdot)$, i.e.

$$E(e^{\theta X_i(t)}) = e^{t\kappa_i(\theta)}, \qquad \theta \leq 0.$$

Clearly, $\kappa_i(\theta)$ is increasing and convex for all $\theta \in \mathbb{R}$ as long as it is well defined. Since we are interested in the light tail behavior, we assume now

(2-i) $\kappa_i(\theta_i^{(0)}) < \infty$ for some $\theta_i^{(0)} > 0$ for $i = 1, 2$.

This implies that $E(X_i(1)) = \kappa_i'(0)$ is positive and finite. Let $\lambda_i = E(X_i(1))$, which is the mean input rate at node i.

Then, according to the decomposition of $X_i(t)$, the exponent $\kappa_i(\cdot)$ can be decomposed as

$$\kappa_i(\theta) = a_i\theta + \frac{1}{2}\sigma_i^2\theta^2 + \kappa_i^{(0)}(\theta) + \kappa_i^{(1)}(\theta), \qquad i = 1, 2, \tag{2.2}$$

where

$$\kappa_i^{(0)}(\theta) = \int_0^1 (e^{\theta x} - 1 - \theta x)\nu_i(dx), \qquad \kappa_i^{(1)}(\theta) = \int_{1+}^\infty (e^{\theta x} - 1)\nu_i(dx).$$

From the definitions,

$$\lambda_i = a_i + E(J_i^{(1)}(1)), \qquad i = 1, 2,$$

so $\lambda_i = a_i$ if the exogenous input at queue i has no jump.

Denote the processing rate at node i by $c_i > 0$. Let $L_i(t)$ be buffer content at node i at time $t \geq 0$ for $i = 1, 2$, which are formally defined as

$$L_1(t) = L_1(0) + X_1(t) - c_1 t + Y_1(t), \tag{2.3}$$
$$L_2(t) = L_2(0) + X_2(t) + c_1 t - Y_1(t) - c_2 t + Y_2(t), \tag{2.4}$$

where $Y_i(t)$ is a regulator at node i, that is, a minimal nondecreasing process for $L_i(t)$ to be nonnegative. Namely, $(L_1(t), L_2(t))$ is a generated by a reflection mapping from net flow processes $(X_1(t) - c_1 t, X_2(t) + c_1 t - c_2 t)$ with reflection matrix

$$R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

See Section 6.3 for R, and the literature [5, 8, 19] for the reflection mapping. Thus, $(L_1(t), L_2(t))$ is adapted to filtration $\mathcal{F}_t^{\boldsymbol{X}}$. We refer to the model described by this reflected process as a Lévy-driven tandem queue.

It is easy to see that this tandem queue has the stationary distribution if and only if

(2-ii) $\lambda_1 < c_1$ and $\lambda_1 + \lambda_2 < c_2$.

We assume this stability condition throughout the paper, and denote the stationary distribution by π.

We consider two types of asymptotic tail behavior of π, called rough and exact asymptotics. Let $g(x)$ a positive valued function of $x \in [0, \infty)$. If

$$\alpha = \lim_{x \to \infty} -\frac{1}{x} \log g(x)$$

exists, $g(x)$ is said to have rough decay rate α. On the other hand, if there exists a function h such that

$$\lim_{x \to \infty} \frac{g(x)}{h(x)} = 1,$$

then $g(x)$ is said to have exact asymptotics $h(x)$. Let (L_1, L_2) be a random vector with distribution π. Our main interest is to find exact asymptotics of $P(d_1 L_1 + d_2 L_2 > x)$ for $\boldsymbol{d} \equiv (d_1, d_2) \geq \boldsymbol{0}$. We are particularly interested in exact asymptotics of the marginal distributions of L_1 and L_2, and denote their rough decay rates by α_1 and α_2, respectively.

Those exact asymptotics will be determined by their moment generating functions (see, e.g., [1]). In the sequel we will use

$$\varphi(\theta_1, \theta_2) = E_\pi(e^{\theta_1 L_1 + \theta_2 L_2}),$$
$$\varphi_1(\theta_2) = E_\pi(\int_0^1 e^{\theta_2 L_2(u)} dY_1(u)), \qquad \varphi_2(\theta_1) = E_\pi(\int_0^1 e^{\theta_1 L_1(u)} dY_2(u)),$$

where φ_i for $i = 1, 2$ are not directly related to φ, but will be useful. It will be shown in Lemma 2.2 that $E_\pi(Y_i(1))$ for $i = 1, 2$ are finite. Using this fact, we can see that

$\varphi_i(\theta_i)/\varphi_i(0)$ is the moment generating function of L_{3-i} under Palm distribution concerning the random measure $\int_B dY_i(u)$ for $B \in \mathcal{B}(\mathbb{R})$, where $Y_1(t)$ and $Y_2(t)$ are extended on the whole line $\mathbb{R}$ under the stationary probability measure for the reflected process generated by the initial distribution π (e.g., see [16] for Palm distribution). Intuitively, it may be considered as the conditional moment generating function of L_{3-i} given that $L_i = 0$. However, it should be noted that $Y_i(t)$ increases on the set of Lebesgue measure 0 if the input has a continuous component.

We start with deriving the stationary equation for the moment generating functions of π. For this, we use Itô's integral formula (see, e.g., Chapter 26 of [11]). Let $C^2(\mathbb{R}^2)$ be the set of all functions from $\mathbb{R}^2$ to $\mathbb{R}$ such that they have continuous second order partial derivatives. Denote $\frac{\partial}{\partial x_i} f(x_1, x_2)$ by $f'_i((x_1, x_2)$, and $\frac{\partial^2}{\partial x_i \partial x_i} f(x_1, x_2)$ by $f''_{ij}((x_1, x_2)$. Then, for $f \in C^2(\mathbb{R}^2)$, applying Itô's integral formula to $f(L_1(t), L_2(t))$ we have

$$\begin{aligned}
f(L_1(t), L_2(t)) - f(L_1(0), L_2(0)) &= \int_0^t f'_1(L_1(u), L_2(u))((a_1 - c_1)du + dB_1(u)) \\
&+ \int_0^t f'_2(L_1(u), L_2(u))((c_1 + a_2 - c_2)du + dB_2(u)) \\
&+ \int_0^t f'_1(0, L_2(u))dY_1(u) - \int_0^t f'_2(0, L_2(u))dY_1(u) + \int_0^t f'_2(L_1(u), 0)dY_2(u) \\
&+ \sum_{i=1}^2 \Big(\int_0^t f'_i(L_2(u), L_2(u))dJ_i^{(0)}(u) + \frac{1}{2}\int_0^t f''_{ii}(L_1(u), L_2(u))\sigma_i^2 du \\
&\quad + \sum_{0<u\le t} (\Delta_i f(L_1(u), L_2(u))1(J_i^{(0)}(u) > 0) - f'_i(L_1(u-), L_2(u-))\Delta J_i^{(0)}(u)) \\
&\quad + \sum_{0<u\le t} \Delta_i f(L_1(u), L_2(u))1(\Delta J_i^{(1)}(u) > 0)\Big),
\end{aligned} \tag{2.5}$$

where we have used the fact that $Y_i(t)$ is increasing if and only if $L_i(t) = 0$, and Δ_i's are defined by

$$\begin{aligned}
\Delta_1 f(L_1(u), L_2(u))) &= f(L_1(u), L_2(u-))) - f(L_1(u-), L_2(u-))), \\
\Delta_2 f(L_1(u), L_2(u))) &= f(L_1(u-), L_2(u))) - f(L_1(u-), L_2(u-))).
\end{aligned}$$

Writing Itô's integral formula (2.5) we used the fact that $Y_i(t)$'s are continuous, which follows from the assumption that $X_i(t)$'s have positive jumps only. Taking the expectation of (2.5) for $t = 1$ given that $(L_1(0), L_2(0))$ is subject to the stationary distribution π, denote this expectation by E_π. Then, as long as all expectations but one below are finite, we have

$$(a_1 - c_1)E_\pi(f_1'(L_1, L_2)) + E_\pi(\int_0^1 f_1'(0, L_2(u))dY_1(u)) - E_\pi(\int_0^1 f_2'(0, L_2(u))dY_1(u))$$
$$+(c_1 + a_2 - c_2)E_\pi(f_2'(L_1, L_2)) + E_\pi(\int_0^1 f_2'(L_1(u), 0)dY_2(u)) + \sum_{i=1}^2 \frac{\sigma_i^2}{2} E_\pi(f_{ii}''(L_1, L_2)))$$
$$+\sum_{i=1}^2 E_\pi(\sum_{0<u\le t} (\Delta_i f(L_1(u), L_2(u))1(\Delta J_i^{(0)} > 0) - f_i'(L_1(u-), L_2(u-))\Delta J_i^{(0)}(u)))$$
$$+\sum_{i=1}^2 E_\pi(\sum_{0<u\le t} \Delta_i f(L_1(u), L_2(u))1(\Delta J_i^{(1)} > 0)) = 0. \tag{2.6}$$

There arise two small technical problems in obtaining (2.6). One is that the expectation of the left-hand side of (2.5) must be finite. However, this can be easily overcome by truncation arguments (see, e.g., [16]), so f, f_i', f_{ij}'' can be unbounded. Another problem is the finiteness of $E_\pi(Y_i(1))$ for $i = 1, 2$. This will be verified in Lemma 2.2 below.

We now derive the basic stationary equation for moment generating function of (L_1, L_2). For this, we prepare two lemmas.

Lemma 2.1 Let $\boldsymbol{L}(u) = (L_1(u), L_2(u))$, and let f be a component-wise differentiable function from $\mathbb{R}_+^2$ to $\mathbb{R}_+$. Then, if $\frac{1}{x^2}(f(\boldsymbol{y} + x\mathbf{e}_i) - f(\boldsymbol{y}) - xf'(\boldsymbol{y}))$ is uniformly bounded for all $x \in (0, 1]$ and $\boldsymbol{y} \ge 0$, then

$$E_\pi(\sum_{0<u\le 1} (\Delta_i f(\boldsymbol{L}(u))1(\Delta J_i^{(0)}(u) > 0) - f_i'(\boldsymbol{L}(u-))\Delta J_i^{(0)}(u)))$$
$$= \int_0^1 E_\pi(f(\boldsymbol{L}(0) + x\mathbf{e}_i) - f(\boldsymbol{L}(0)) - f'(\boldsymbol{L}(0))x)\nu_i(dx), \tag{2.7}$$

and, if $\frac{1}{x}(f(\boldsymbol{y} + x\mathbf{e}_i) - f(\boldsymbol{y}))$ is uniformly bounded for all $x > 1$ and $\boldsymbol{y} \ge 0$, then

$$E_\pi(\sum_{0<u\le 1} \Delta_i f(\boldsymbol{L}(u))1(\Delta J_i^{(1)}(u) > 0)) = \int_1^\infty E_\pi(f(\boldsymbol{L}(0) + x\mathbf{e}_i) - f(\boldsymbol{L}(0)))\nu_i(dx), \tag{2.8}$$

where $\mathbf{e}_1 = (1, 0)$ and $\mathbf{e}_2 = (0, 1)$.

Proof. From the definition of $L_i(u)$ and Λ_i, we have

$$E_\pi\Big(\sum_{0<u\le 1} (\Delta_i f(\boldsymbol{L}(u))1(\Delta J_i^{(0)}(u) > 0) - f_i'(\boldsymbol{L}(u-))\Delta J_i^{(0)}(u))\Big)$$
$$= E_\pi\Big(\int_0^1 \int_0^1 (f(\boldsymbol{L}(u-) + x\mathbf{e}_i) - f(\boldsymbol{L}(u-)) - xf_i'(\boldsymbol{L}(u-)))\Lambda_i(du, dx)\Big)$$
$$= E_\pi\Big(\int_0^1 \int_0^1 E(f(\boldsymbol{L}(u-) + x\mathbf{e}_i) - f(\boldsymbol{L}(u-)) - xf_i'(\boldsymbol{L}(u-))|\mathcal{F}_{u-})du\nu_i(dx)\Big)$$
$$= \int_0^1 \int_0^1 E_\pi\Big(f(\boldsymbol{L}(u-) + x\mathbf{e}_i) - f(\boldsymbol{L}(u-)) - xf_i'(\boldsymbol{L}(u-))\Big)du\nu_i(dx)$$
$$= \int_0^1 E_\pi\Big(f(\boldsymbol{L}(0) + x\mathbf{e}_i) - f(\boldsymbol{L}(0)) - xf_i'(\boldsymbol{L}(0))\Big)\nu_i(dx),$$

where the third equality is obtained by the Fubini's theorem due to the uniformly bounded assumption. Similarly, (2.8) is obtained. □

Lemma 2.2 Under the conditions (2-i) and (2-ii),

$$E_\pi(Y_1(1)) = c_1 - \lambda_1, \tag{2.9}$$
$$E_\pi(Y_2(1)) = c_2 - (\lambda_1 + \lambda_2). \tag{2.10}$$

PROOF. Let $f(x_1, x_2) = x_1$ in (2.6), then we get (2.9) using Lemma 2.1 since all terms but $E_\pi(Y_1(1))$ in the left-hand side of (2.6) are finite. Similarly, letting $f(x_1, x_2) = x_2$ yields

$$(c_1 - c_2) - E_\pi(Y_1(1)) + E_\pi(Y_2(1)) = 0.$$

Hence, we have (2.10). □

Remark 2.1 One may think that (2.9) and (2.10) are immediate from (2.3) and (2.4) by taking the expectation under π. However, this requires the finiteness of $E_\pi(L_1)$ and $E_\pi(L_2)$, which we cannot use at this stage.

Proposition 2.1 For $\theta_1, \theta_2 \le 0$ we have

$$\gamma(\theta_1, \theta_2)\varphi(\theta_1, \theta_2) = (\theta_1 - \theta_2)\varphi_1(\theta_2) + \theta_2\varphi_2(\theta_1), \tag{2.11}$$

where

$$\gamma(\theta_1, \theta_2) = c_1\theta_1 + (c_2 - c_1)\theta_2 - \kappa_1(\theta_1) - \kappa_2(\theta_2).$$

PROOF. Substituting $f(x_1, x_2) = e^{\theta_1 x_1 + \theta_2 x_2}$ into (2.6) we obtain

$$\begin{aligned}
&(a_1 - c_1)\theta_1 E_\pi(e^{\theta_1 L_1 + \theta_2 L_2}) + \theta_1 E_\pi(\int_0^1 e^{\theta_2 L_2(u)} dY_1(u)) - \theta_2 E_\pi(\int_0^1 e^{\theta_2 L_2(u)} dY_1(u)) \\
&\quad + (c_1 + a_2 - c_2)\theta_2 E_\pi(e^{\theta_1 L_1 + \theta_2 L_2}) + \theta_2 E_\pi(\int_0^1 e^{\theta_1 L_1(u)} dY_2(u)) \\
&\quad + \sum_{i=1}^2 \frac{\sigma_i^2}{2}\theta_i^2 E_\pi(e^{\theta_1 L_1 + \theta_2 L_2}) \\
&\quad + \sum_{i=1}^2 E_\pi\Big(\sum_{0<u\le 1} e^{\theta_1 L_1(u-) + \theta_2 L_2(u-)}(e^{\theta_i \Delta_i J_i^{(0)}(u)} - 1 - \theta_i \Delta_i J_i^{(0)}(u))\Big) \\
&\quad + \sum_{i=1}^2 E_\pi\Big(\sum_{0<u\le 1} e^{\theta_1 L_1(u-) + \theta_2 L_2(u-)}(e^{\theta_i \Delta_i J_i^{(1)}(u)} - 1)\Big).
\end{aligned}$$

We apply Lemma 2.1 to this equation, then, using the definition of the Lévy exponent $\kappa_i(\theta_i)$, we get (2.11). □

Remark 2.2 Equation (2.11) can be directly obtained by applying the Kella-Whitt martingale of [10]. However, Itô's integral formula is more flexible for our arguments.

In general, it is hard to get the stationary distribution π or its moment generating function in closed form from (2.6). There is one special case that φ is obtained in closed form. This is the case that $X_2(t) \equiv 0$, that is, there is no intermediate input. This case has been recently studied in [6]. We revisit it as an example for (2.6), which shows how (2.6) can be used to get useful information.

Example 2.1 (Tandem queue without an intermediate input) Suppose $X_2(t) \equiv 0$ in the Lévy-driven tandem queue satisfying the stability condition (2-ii). We assume that $c_1 > c_2$ since $L_2(t) \equiv 0$ otherwise.

Since $L_2(u) = 0$ implies $L_1(u) = 0$, so $L_1(u) = 0$ when $Y_2(u)$ is increasing, we have $\varphi_2(\theta_1) = E_\pi Y_2(1) = c_2 - \lambda_1$ from Lemma 2.2. Hence by Proposition 2.1 (with $\lambda_2 = 0$, and $\kappa_2(\theta) = 0$) we have

$$(c_1\theta_1 - (c_1 - c_2)\theta_2 - \kappa_1(\theta_1))\varphi(\theta_1, \theta_2) = (\theta_1 - \theta_2)\varphi_1(\theta_2) + (c_2 - \lambda_1)\theta_2. \tag{2.12}$$

Since $\varphi_1(0) = c_1 - \lambda_1$ by (2.9), letting $\theta_2 = 0$ in (2.12) implies the well known Pollaczek-Khinchine formula:

$$E(e^{\theta_1 L_1}) = \varphi(\theta_1, 0) = \frac{(c_1 - \lambda_1)\theta_1}{c_1\theta_1 - \kappa_1(\theta_1)}, \qquad \theta_1 \le 0. \tag{2.13}$$

Assume that $c_1\theta_1 - \kappa_1(\theta_1) = 0$ has a positive solution, which must be the rough decay rate α_1. Furthermore, it is not hard to see that $P(L_1 > x)$ has exact asymptotic $ce^{-\alpha_1 x}$ with known constant c.

We next let $\theta_1 = 0$ in (2.12), then we have

$$\varphi_1(\theta_2) = (c_1 - c_2)\varphi(0, \theta_2) + c_2 - \lambda_1. \tag{2.14}$$

Substituting this into (2.12), we have

$$(c_1\theta_1 - (c_1 - c_2)\theta_2 - \kappa_1(\theta_1))\varphi(\theta_1, \theta_2) = (c_1 - c_2)(\theta_1 - \theta_2)\varphi(0, \theta_2) + (c_2 - \lambda_1)\theta_1. \tag{2.15}$$

For each $\theta_2 \le 0$, let $\xi_1(\theta_2)$ be the smallest solution θ_1 of the equation

$$c_1\theta_1 - \kappa_1(\theta_1) = (c_1 - c_2)\theta_2,$$

which always exists and is negative since $\kappa_1(0) = 0$ and $c_1 - \kappa_1'(0) = c_1 - \lambda_1 > 0$. Then, letting $\theta_1 = \xi_1(\theta_2)$ in (2.15) yields

$$\varphi(0, \theta_2) = \frac{(c_2 - \lambda_1)\xi_1(\theta_2)}{(c_1 - c_2)(\theta_2 - \xi_1(\theta_2))} \tag{2.16}$$

Plugging this into (2.15), we arrive at

$$\varphi(\theta_1, \theta_2) = \frac{(c_2 - \lambda_1)(\theta_1 - \xi_1(\theta_2))\theta_2}{(\theta_2 - \xi_1(\theta_2))(c_1\theta_1 - (c_1 - c_2)\theta_2 - \kappa_1(\theta_1))}. \tag{2.17}$$

This is the formula obtained in [6]. Based on it, asymptotic behavior of the stationary distribution π is studied in [13]. We will extend (2.17) to the case of n nodes in Section 6.3. □

3 Domain of the MGF

We now consider the Lévy-driven tandem queue with the intermediate input. In the stationary equation (2.6), we may rewrite function $\gamma(\theta_1, \theta_2)$ in the form:

$$\gamma(\theta_1, \theta_2) = c_1\theta_1 + (c_2 - c_1)\theta_2 - \kappa_1(\theta_1) - \kappa_2(\theta_2). \tag{3.1}$$

To avoid complicated presentation, we assume in this and next sections that there are no jump inputs at nodes 1 and 2, that is, $J_1(t) = J_2(t) \equiv 0$. In this case, the tandem queue is referred to as a Brownian tandem queue with an intermediate input. We will include those jump inputs in Section 5. The main purpose of this section is to identify the domain of φ

$$\mathcal{D} = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \varphi(\theta_1, \theta_2) < \infty\}.$$

The knowledge of the domain allows us to study asymptotic decay of some interesting tail distributions. We first note that $\mathcal{D}$ is convex since φ is a convex function, and it obviously includes the set $\{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1, \theta_2 \le 0\}$.

Since there are no jump inputs here, (3.1) is simplified to

$$\gamma(\theta_1, \theta_2) = r_1\theta_1 + r_2\theta_2 - \frac{1}{2}(\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2), \tag{3.2}$$

and condition (2-i) is always satisfied, where

$$r_1 = c_1 - \lambda_1, \qquad r_2 = c_2 - c_1 - \lambda_2.$$

Furthermore, $r_1, r_1 + r_2 > 0$ by the stability condition (2-ii), but r_2 can be negative or positive.

Note that the stationary equation (2.11) of Proposition 2.1 holds as long as $\varphi(\theta_1, \theta_2)$, $\varphi_2(\theta_1)$ and $\varphi_1(\theta_2)$ are finite. Hence, $\gamma(z_1, z_2)\varphi(z_1, z_2)$ is an analytic function of two complex variables z_1, z_2 for $\Re z_1 < 0$ and $\Re z_2 < 0$, and this domain is extendable as long as $\varphi_2(z_1)$ and $\varphi_1(z_2)$ are finite, where a complex valued function of two complex variables is said to be analytic if it is analytic as a one variable function for each fixed other variable (see, e.g., II.15 of [15]). Hence, we have proved the following fact.

Lemma 3.1 If both of $\varphi_2(\theta_1)$ and $\varphi_1(\theta_2)$ are finite, then $\gamma(\theta_1, \theta_2)\varphi(\theta_1, \theta_2)$ is finite. In particular, if $\gamma(\theta_1, \theta_2) \neq 0$ in this case, then $\varphi(\theta_1, \theta_2)$ is finite. Conversely, if $\varphi(\theta_1, \theta_2)$ is finite, then $\varphi_2(\theta_1)$ and $\varphi_1(\theta_2)$ are finite.

Using r_1 and r_2, (2.9) and (2.10) of Lemma 2.2 are written as

$$E_\pi(Y_1(1)) = r_1, \qquad E_\pi(Y_2(1)) = r_1 + r_2. \tag{3.3}$$

Since $\varphi_2(0) = E_\pi(Y_2(1))$, substituting $\theta_1 = 0$ in (2.11) yields

$$\varphi_1(\theta_2) = (\frac{1}{2}\sigma_2^2\theta_2 - r_2)\varphi(0, \theta_2) + r_1 + r_2. \tag{3.4}$$

Note that both sides of (3.4) are simultaneously finite or infinite due to Lemma 3.1. Furthermore, $z_2 = \frac{2r_2}{\sigma_2^2}$ is a removable singular point of $\varphi(0, z)$ since

$$\varphi(0, \theta_2) = \frac{\varphi_1(\theta_2) - \varphi_1(z_2)}{\frac{1}{2}\sigma_2^2\theta_2 - r_2}.$$

Hence, we have

Lemma 3.2 $\varphi_1(z)$ and $\varphi(0, z)$ have the same singularity.

We cannot get a similar direct relation between $\varphi_2(\theta_1)$ and $\varphi(\theta_1, 0)$, but the following result will be sufficient. Recall that α_1 is the rough decay rate of L_1.

Lemma 3.3 $\varphi_2(\theta)$ is finite for $\theta < \alpha_1$, where $\alpha_1 = \frac{2r_1}{\sigma_1^2}$.

Remark 3.1 This result will be sharpened in Corollary 3.1.

PROOF. $\alpha_1 = \frac{2r_1}{\sigma_1^2}$ is immediate from Example 2.1 since the first queue is unchanged by the intermediate input. Let $\Gamma_0 = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \gamma(\theta_1, \theta_2) > 0, \theta_1 < \alpha_1\}$. Since Γ_0 is an open convex set, we can find $(\theta_1^{(\epsilon)}, \theta_2^{(\epsilon)}) \in \Gamma_0$ for any $\epsilon > 0$ such that $\max(0, \alpha_1 - \epsilon) < \theta_1^{(\epsilon)} < \alpha_1$ and $\theta_2^{(\epsilon)} < 0$. Substituting this $(\theta_1^{(\epsilon)}, \theta_2^{(\epsilon)})$ into (2.11), we have

$$-\theta_2^{(\epsilon)}\varphi_1(\theta_2^{(\epsilon)}) + \gamma(\theta_1^{(\epsilon)}, \theta_2^{(\epsilon)})\varphi(\theta_1^{(\epsilon)}, \theta_2^{(\epsilon)}) = (\theta_1^{(\epsilon)} - \theta_2^{(\epsilon)})\varphi_2(\theta_1^{(\epsilon)}).$$

Since all the coefficients of $\varphi(\theta_1^{(\epsilon)}, \theta_2^{(\epsilon)}), \varphi_1(\theta_2^{(\epsilon)})$ and $\varphi_2(\theta_1^{(\epsilon)})$ are positive and $\varphi_1(\theta_2^{(\epsilon)}) < \varphi_1(0) = r_1 < \infty$, $\varphi_2(\theta_1^{(\epsilon)})$ must be finite. This proves the lemma since $\varphi_2(\theta)$ is increasing and ϵ can be arbitrarily small. □

We next rewrite (2.11) as

$$\gamma(\theta_1, \theta_2)\varphi(\theta_1, \theta_2) + (\theta_2 - \theta_1)\varphi_1(\theta_2) = \theta_2\varphi_2(\theta_1). \tag{3.5}$$

Since both sides of (3.5) are simultaneously finite or infinite, similarly to Lemma 3.1, it follows from Lemma 3.3 that $\varphi(\theta_1, \theta_2)$ and $\varphi_1(\theta_2)$ must be positive and finite for (θ_1, θ_2) in the region:

$$\mathcal{D}_+^{(1)} = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1, \gamma(\theta_1, \theta_2) > 0, 0 < \theta_1 < \theta_2\}.$$

Since $\gamma(\theta_1, \theta_2) = 0$ is an ellipse, we let

$$(\theta_1^{\max}, \theta_2^{\max}) = \arg\max_{(\theta_1,\theta_2)}\{\theta_2; \gamma(\theta_1, \theta_2) = 0\}, \quad (\theta_1^{\min}, \theta_2^{\min}) = \arg\min_{(\theta_1,\theta_2)}\{\theta_2; \gamma(\theta_1, \theta_2) = 0\},$$
$$(\eta_1^{\max}, \eta_2^{\max}) = \arg\max_{(\theta_1,\theta_2)}\{\theta_1; \gamma(\theta_1, \theta_2) = 0\}, \quad (\eta_1^{\min}, \eta_2^{\min}) = \arg\min_{(\theta_1,\theta_2)}\{\theta_1; \gamma(\theta_1, \theta_2) = 0\},$$
$$\beta = \max\{\theta; \gamma(\theta, \theta) = 0\}.$$

It is easy to compute these values. For example, $\theta_1^{\max} = \frac{1}{2}\alpha_1 < \alpha_1$, and solving $\gamma(\beta, \beta) = 0$ we have

$$\beta = \frac{2(r_1 + r_2)}{\sigma_1^2 + \sigma_2^2}, \tag{3.6}$$

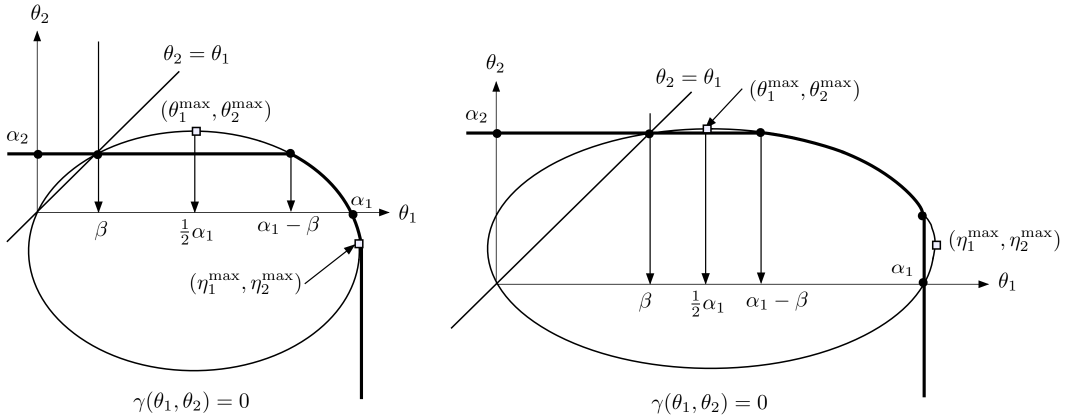
where $\beta > 0$ by the stability condition (2-ii). However, we will not use these specific values as long as possible for applying our arguments to more general Lévy inputs. We next let

$$(\beta_1^{\max}, \beta_2^{\max}) = \arg \max_{(\theta_1,\theta_2)} \{\theta_2; (\theta_1, \theta_2) \in \mathcal{D}_+^{(1)}\}.$$

It is easy to see from the definition of $\mathcal{D}_+^{(1)}$ that

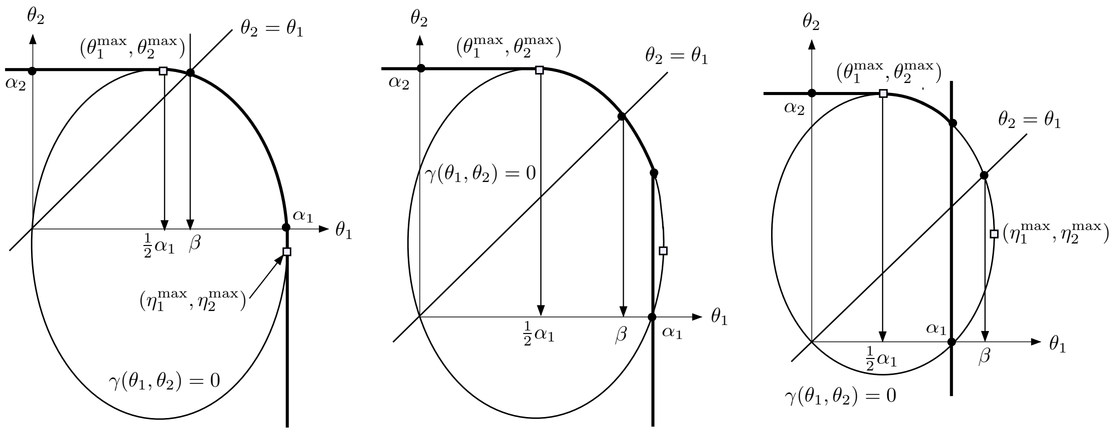
$$\beta_2^{\max} = \begin{cases} \beta, & \beta < \frac{1}{2}\alpha_1, \\ \theta_2^{\max}, & \frac{1}{2}\alpha_1 \le \beta, \end{cases} \tag{3.7}$$

and $\varphi_1(\theta)$ is finite for $\theta < \beta_2^{\max}$ (see Figures 1 and 2).



(a$-$) $\beta < \frac{1}{2}\alpha_1$ with $r_2 < 0$ (a+) $\beta < \frac{1}{2}\alpha_1$ with $r_2 > 0$

Figure 1: Typical regions of $\mathcal{D}^{\mathrm{o}}$ for $\beta < \frac{1}{2}\alpha_1$



(b$-$) $\frac{1}{2}\alpha_1 \le \beta \le \alpha_1$ with $r_2 < 0$ (b+) $\frac{1}{2}\alpha_1 \le \beta \le \alpha_1$ with $r_2 > 0$ (c) $\alpha_1 < \beta$

Figure 2: Typical regions of $\mathcal{D}^{\mathrm{o}}$ for $\frac{1}{2}\alpha_1 \le \beta$

Having this $\beta_2^{\max}$ in mind, we define $\mathcal{D}_+^{(2)}$ as

$$\mathcal{D}_+^{(2)} = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1, \theta_2 < \beta_2^{\max}, \gamma(\theta_1, \theta_2) > 0, 0 \le \theta_2 \le \theta_1\},$$

and consider (2.11). Since $\varphi_2(\theta_1)$ and $\varphi_1(\theta_2)$ are finite for $(\theta_1,\theta_2) \in \mathcal{D}_+^{(2)}$, the right-hand side of (2.11) is finite and therefore $\varphi(\theta_1,\theta_2)$ must be positive and finite for $(\theta_1,\theta_2) \in \mathcal{D}_+^{(2)}$. Let $\mathcal{D}_+ = \mathcal{D}_+^{(1)} \cup \mathcal{D}_+^{(2)}$. Since $(\theta_1,\theta_2) \in \mathcal{D}_+^{(1)}$ implies $\theta_2 < \beta_2^{\max}$, we have

$$\mathcal{D}_+ = \{(\theta_1,\theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1, \theta_2 < \beta_2^{\max}, \gamma(\theta_1,\theta_2) > 0, \theta_2 \geq 0\}.$$

We finally rewrite (2.11) as

$$\gamma(\theta_1,\theta_2)\varphi(\theta_1,\theta_2) - \theta_2\varphi_2(\theta_1) = (\theta_1 - \theta_2)\varphi_1(\theta_2), \tag{3.8}$$

and define $\mathcal{D}_-$ as

$$\mathcal{D}_- = \{(\theta_1,\theta_2) \in \mathbb{R}^2; \gamma(\theta_1,\theta_2) > 0, 0 \leq \theta_2 \leq \theta_1, \theta_2 < 0\}.$$

Since $\varphi_1(\theta_2)$ is finite for $\theta_2 < 0$, $\varphi(\theta_1,\theta_2)$ must be finite for $(\theta_1,\theta_2) \in \mathcal{D}_-$ similarly to the previous cases.

We are now in a position to identify $\mathcal{D}$ except for its boundary. Let $\xi_1(\theta_2)$ be the minimal solution θ_1 for $\gamma(\theta_1,\theta_2) = 0$ for each θ_2, and let $\xi_2(\theta_1)$ be the maximal solution θ_2 for $\gamma(\theta_1,\theta_2) = 0$ for each θ_1, as long as they exists. Clearly, for $\eta_2^{\min} \leq \theta_2 \leq \theta_2^{\max}$, $\theta_1 = \xi_1(\theta_2)$ if and only if $\theta_2 = \xi_2(\theta_1)$.

Proposition 3.1 For the Brownian tandem queue with an intermediate input satisfying the condition (2-ii), let $\mathcal{D}^{\mathrm{o}}$ be interior of $\mathcal{D}$. Then,

$$\mathcal{D}^{\mathrm{o}} = \{(\theta_1,\theta_2) \in \mathbb{R}^2; (\theta_1,\theta_2) < (\theta_1',\theta_2') \text{ for some } (\theta_1',\theta_2') \in \mathcal{D}_+ \cup \mathcal{D}_-\}. \tag{3.9}$$

Proof. Denote the right-hand side of (3.9) by $\mathcal{A}$. We have already proved that $\mathcal{D}_+ \cup \mathcal{D}_- \subset \mathcal{D}$, which implies $\mathcal{A} \subset \mathcal{D}$. So, we only need to prove that $\varphi(\theta_1,\theta_2) = \infty$ if $(\theta_1,\theta_2) \notin \overline{\mathcal{A}}$, where $\overline{\mathcal{A}}$ is the closure of $\mathcal{A}$. We consider the following three cases separately.

If $\beta < \frac{1}{2}\alpha_1$, then $(\beta,\beta) \in \overline{\mathcal{A}}$ (see Figure 1). Hence, $\varphi(\theta_1,\theta_2) < \infty$ for $\theta_1,\theta_2 < \beta$, and

$$\varphi_1(\theta_2) = \frac{\theta_2\varphi_2(\xi_1(\theta_2))}{\theta_2 - \xi_1(\theta_2)}, \qquad 0 < \theta_2 < \beta. \tag{3.10}$$

Since $\beta = \xi_1(\beta) < \alpha_1$, $\varphi_1(z)$ has a simple pole at $z = \beta$. By Lemma 3.2, $\varphi(0,z)$ has the same pole at $z = \beta$. Hence, $\theta_2 > \beta$ and $\theta_1 \geq 0$ imply $\varphi(\theta_1,\theta_2) = \infty$.

If $\frac{1}{2}\alpha_1 \leq \beta$, then $(\theta_1^{\max},\theta_2^{\max}) \in \overline{\mathcal{A}}$ (see Figure 2). We rearrange (3.5) as

$$(\theta_2 - \theta_1)\varphi_1(\theta_2) = -\gamma(\theta_1,\theta_2)\varphi(\theta_1,\theta_2) + \theta_2\varphi_2(\theta_1).$$

Choose any point $(\theta_1,\theta_2) \in \overline{\mathcal{A}}^c \cap \{(\theta_1,\theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1\}$, and assume that $\varphi(\theta_1,\theta_2) < \infty$. Since $\varphi_2(\theta_1)$ is finite, $\varphi_1(\theta_2)$ must be finite, which is proved by partial differentiation with respect to θ_1. Since the left-hand side is always finite for any θ_1, we let θ_1 go to θ_2 or increase to α_1. Both leads to contradiction, and we have $\varphi(\theta_1,\theta_2) = \infty$. If $\beta < \frac{1}{2}\alpha_1$, we can similarly prove that $(\theta_1,\theta_2) \in \overline{\mathcal{A}}^c \cap \{(\theta_1,\theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1\}$ implies $\varphi(\theta_1,\theta_2) = \infty$.

Obviously, if $\theta_1 > \alpha_1$ and $\theta_2 \geq 0$, then $\varphi(\theta_1,\theta_2) = \infty$ while, if $\theta_1 \leq \alpha_1$ and $\theta_2 \leq 0$, then $\varphi(\theta_1,\theta_2) < \infty$. Furthermore, $\theta_1 < \eta_1^{\max}$ and $\theta_2 < \eta_2^{\max}$ imply $\varphi(\theta_1,\theta_2) < \infty$ from the definition of $\mathcal{A}$. Thus, it remains to prove that $(\theta_1,\theta_2) \in \overline{\mathcal{A}}^c \cap \{(\theta_1,\theta_2) \in \mathbb{R}^2; \alpha_1 <$

$\theta_1, \eta_2^{\max} < \theta_2 < 0\}$ implies $\varphi(\theta_1,\theta_2) = \infty$. For this (θ_1,θ_2), assume that $\varphi(\theta_1,\theta_2) < \infty$. From the definition of $\mathcal{A}$, $\gamma(\theta_1,\theta_2) < 0$, and $\varphi_1(\theta_2) < \infty$ for $\theta_2 < 0$. Hence, rearranging (3.5) as

$$-\gamma(\theta_1,\theta_2)\varphi(\theta_1,\theta_2) + (\theta_1 - \theta_2)\varphi_1(\theta_2) = -\theta_2\varphi_2(\theta_1),$$

we can see that $\varphi_2(\theta_1)$ must be finite. Let negative θ_2 goes to zero, then the left-hand side must be positive while the right hand side vanishes. This implies that $\varphi_2(\theta_1)$ cannot be finite. Thus, we have a contradiction, and the proof is completed. □

The following corollary sharpens Lemma 3.3.

Corollary 3.1 Let $\alpha_1^{\max} = \sup\{\theta \geq 0; \varphi_2(\theta) < \infty\}$. Then,

$$\alpha_1^{\max} = \begin{cases} \alpha_1, & \eta_2^{\max} \geq 0, \\ \eta_1^{\max}, & \eta_2^{\max} < 0, \end{cases} \tag{3.11}$$

Proof. From (3.5),

$$-\gamma(\theta_1,\theta_2)\varphi(\theta_1,\theta_2) = (\theta_1 - \theta_2)\varphi_1(\theta_2) - \theta_2\varphi_2(\theta_1),$$

This implies that, if $\theta_2 < \beta_2^{\max}$, then $\varphi(\theta_1,\theta_2) < \infty$ if and only if $\varphi_2(\theta_1) < \infty$. Since $(\theta_1,\theta_2) \in \mathcal{D}^{\mathrm{o}}$ implies $\theta_2 < \beta_2^{\max}$, $\alpha_1^{\max} = \sup\{\theta_1; (\theta_1,\theta_2) \in \mathcal{D}^{\mathrm{o}}\}$. Hence, Proposition 3.1 concludes (3.11). □

We can write (3.9) in a more explicit form. For this, let

$$\mathcal{D}_1 = \{(\theta_1,\theta_2) \in \mathbb{R}^2; \theta_1 < \alpha_1^{\max}, \theta_2 < \beta_2^{\max}\},$$
$$\mathcal{D}_2 = \{(\theta_1,\theta_2) \in \mathbb{R}^2; \theta_1 < \theta_1', \theta_2 < \theta_2', \text{ for some } (\theta_1',\theta_2') \text{ such that } \gamma(\theta_1',\theta_2') \leq 0\},$$

then it is easy to get the following corollary from Proposition 3.1 and Corollary 3.1.

Corollary 3.2 $\mathcal{D}^{\mathrm{o}} = \mathcal{D}_1 \cap \mathcal{D}_2$, and $\mathcal{D}^{\mathrm{o}}$ is a convex set.

The open domain $\mathcal{D}^{\mathrm{o}}$ is typical for the discrete-time two dimensional reflected process on the quadrant, but does not cover all the cases because of the structure of a tandem queue. In the terminology of the sample path large deviations, it is important to find the optimal path for the rate function in each direction. For the direction to increase L_2, this path goes up along the 2nd coordinate in Figure 1 while it straightly moves inside the quadrant in cases (b) and (c). So, we do not have the case that the optimal path firstly goes along the 1st coordinate, then straightly move inside the quadrant.

4 Exact asymptotic behavior

In this section, we first derive the exact asymptotics of the tail probability of L_2. For this, we will study the type of singularity of $\varphi(0,\theta)$ at the boundary of $\mathcal{D}$. In a similar way, we work out the exact asymptotics of the tail distribution function of $d_1L_1 + d_2L_2$ for $d_1, d_2 > 0$. We follow the idea from [1, 13], which is based on the Heaviside's operational principle. Thus suppose that $\psi(\theta) = \int_0^\infty e^{\theta x}\overline{F}(x)\,dx$ has the leftmost singularity α. Then, by Tauberian theorem for ultimately monotone density (e.g., see Section XIII.5 of [7]), applied to $\psi(-\theta + \alpha)$, (S1) if, for $s > 0$,

$$\lim_{\theta \uparrow \alpha} (\alpha - \theta)^s \psi(\theta) = C_1', \qquad \theta \uparrow \alpha,$$

then

$$\overline{F}(x) = \frac{C_1'}{\Gamma(s)} x^{s-1} e^{-\alpha x}(1 + o(1)),$$

where $\Gamma(s)$ is the gamma function, while, by the Heaviside's operational principle,

(S2) if for some constant K and non integer $s > 0$

$$\psi(\theta) = K - C_2'(\alpha - \theta)^s + o((\alpha - \theta)^s), \qquad \theta \uparrow \alpha$$

then

$$\overline{F}(x) = \frac{C_2'}{\Gamma(-s)} x^{-s-1} e^{-\alpha x}(1 + o(1)) \ .$$

Theorem 4.1 For the Brownian tandem queue satisfying the stability condition (2-ii), $P(L_2 > x)$ has the exact asymptotics $h(x)$ of the following type.

(4a) If $\beta < \frac{1}{2}\alpha_1$, then $h(x) = C_1 e^{-\beta x}$.

(4b) If $\frac{1}{2}\alpha_1 = \beta$, then $h(x) = C_2 x^{-\frac{1}{2}} e^{-\theta_2^{\max} x}$.

(4c) If $\frac{1}{2}\alpha_1 < \beta$, then $h(x) = C_3 x^{-\frac{3}{2}} e^{-\theta_2^{\max} x}$.

Constants C_1, C_2 and C_3 are given in the proof. Hence, the rough decay rate α_2 is identical with $\beta_2^{\max}$ of (3.7)

Remark 4.1 Quantities α_1, β and $\theta_2^{\max}$ are defined in Section 3. Specifically they are given by

$$\alpha_1 = \frac{2r_1}{\sigma_1^2}, \qquad \beta = \frac{2(r_1 + r_2)}{\sigma_1^2 + \sigma_2^2}, \qquad \theta_2^{\max} = \frac{1}{\sigma_2^2}\left(r_2 + \sqrt{r_2^2 + r_1^2 \frac{\sigma_2^2}{\sigma_1^2}} \right).$$

For the proof of Theorem 4.1, we consider $\psi(\theta) = \int_0^\infty e^{\theta x} P(L_2 > x)\, dx$. Since $\varphi(0, \theta) = 1 + \theta\psi(\theta)$, it follows from (3.4), (3.10) and Proposition 3.1 that, for $\theta_2 < \alpha_2$,

$$\begin{aligned} \psi(\theta_2) &= \frac{\varphi(0, \theta_2) - 1}{\theta_2} \\ &= \frac{\varphi_1(\theta_2) - (r_1 + r_2)}{\theta_2 f(\theta_2)} - \frac{1}{\theta_2} \\ &= \frac{\varphi_2(\xi_1(\theta_2))}{f(\theta_2)(\theta_2 - \xi_1(\theta_2))} - \frac{\frac{1}{2}\sigma_2^2\theta_2 + r_2}{\theta_2 f(\theta_2)}, \end{aligned} \tag{4.1}$$

where $f(\theta) = \frac{1}{2}\sigma_2^2\theta - r_2$.

For cases (4b) and (4c), we prepare the following lemma.

Lemma 4.1 For $\theta_2 \uparrow \theta_2^{\max}$,

$$\theta_1^{\max} - \xi_1(\theta_2) = \sqrt{\frac{2(\theta_2^{\max} - \theta_2)}{-\xi_2''(\theta_1^{\max})}} + o(|\theta_2^{\max} - \theta_2|^{\frac{1}{2}}), \tag{4.2}$$

and, particularly if $\beta = \theta_2^{\max}$, then

$$\theta_2 - \xi_1(\theta_2) = \sqrt{\frac{2(\theta_2^{\max} - \theta_2)}{-\xi_2''(\theta_1^{\max})}} + o(|\theta_2^{\max} - \theta_2|^{\frac{1}{2}}), \tag{4.3}$$

where

$$\xi_2''(\theta_1^{\max}) = -\frac{\sigma_1^3}{\sqrt{r_1^2\sigma_1^2 + r_2^2\sigma_2^2}} < 0.$$

Proof. Since $\xi_2(\theta_1)$ is concave from its definition and $\xi_2'(\theta_1^{\max}) = 0$, its Taylor expansion at $\theta_1 = \theta_1^{\max}$ yields

$$\xi_2(\theta_1) = \xi_2(\theta_1^{\max}) + \frac{1}{2}\xi_2''(\theta_1^{\max})(\theta_1 - \theta_1^{\max})^2 + o((\theta_1 - \theta_1^{\max})^2),$$

which implies, for $\eta_2^{\min} < \theta_2 < \theta_2^{\max}$, or equivalently, $\eta_1^{\min} < \theta_1 < \theta_1^{\max}$,

$$\theta_1^{\max} - \theta_1 = \sqrt{\frac{2(\xi_2(\theta_1^{\max}) - \xi_2(\theta_1))}{-\xi_2''(\theta_1^{\max})}} + o(|\theta_1 - \theta_1^{\max}|). \tag{4.4}$$

Since $\theta_2 = \xi_2(\theta_1)$ is equivalent to $\theta_1 = \xi_1(\theta_2)$ for $\eta_2^{\min} < \theta_2 < \theta_2^{\max}$, this can be written as

$$\theta_1^{\max} - \xi_1(\theta_2) \;=\; \sqrt{\frac{2(\theta_2^{\max} - \theta_2)}{-\xi_2''(\theta_1^{\max})}} + o(|\theta_2 - \theta_2^{\max}|^{\frac{1}{2}}).$$

Hence, we have (4.2). If $\beta = \theta_2^{\max}$, then $\theta_1^{\max} = \theta_2^{\max}$, so we have

$$\theta_2 - \xi_1(\theta_2) \;=\; \theta_1^{\max} - \xi_1(\theta_2) - (\theta_2^{\max} - \theta_2) + o(|\theta_1 - \theta_1^{\max}|^2).$$

This and (4.2) yield (4.3). It remains to compute $\xi_2''(\theta_1^{\max})$, but this is easily done by differentiating $\gamma(\theta, \theta') = 0$ with respect to θ at $(\theta, \theta') = (\theta_1^{\max}, \theta_2^{\max})$. □

The proof of Theorem 4.1 We consider the singularity of $\psi(z)$, which is the same as $\varphi(0, z)$, and therefore the same as $\varphi_1(z)$ by Lemma 3.2. We prove the three cases separately.

For (4a), we assume that $\beta < \frac{1}{2}\alpha_1$. In this case, we have already observed in the proof of Proposition 3.1 that $\varphi_1(z)$ has a simple pole at $z = \beta$ (see (3.10)), so $\psi(z)$ also has the same pole by the above singularity arguments. Hence, (S1) with $s = 1$ leads to (4a). The constant C_1 is computed from (4.1) as,

$$\begin{aligned} C_1 &= \lim_{\theta \to \beta} (\beta - \theta)\psi(\theta) \\ &= \lim_{\theta \to \beta} \frac{(\beta - \theta)}{\theta(\theta - \xi_1(\theta))} \frac{\theta\varphi_2(\xi_1(\theta)) - (r_1 + r_2)(\theta - \xi_1(\theta))}{f(\theta)} \\ &= \frac{-1}{\beta(1 - \xi_1'(\beta))} \lim_{\theta \to \beta} \frac{\theta\varphi_2(\xi_1(\theta)) - (r_1 + r_2)(\theta - \xi_1(\theta))}{f(\theta)}. \end{aligned}$$

The condition $\beta < \frac{1}{2}\alpha_1$ is equivalent to

$$r_1\sigma_2^2 - r_1\sigma_1^2 - (r_1 + r_2)\sigma_1^2 > 0.$$

This implies that $f(\beta) = \frac{1}{2}\sigma_2^2\beta - r_2 = \frac{r_1\sigma_2^2 - r_2\sigma_1^2}{\sigma_1^2+\sigma_2^2} > 0$. From (A.1) of Appendix A,

$$\xi_1'(\beta) = 1 + \frac{(r_1 + r_2)(\sigma_1^2 + \sigma_2^2)}{r_1\sigma_2^2 - r_1\sigma_1^2 - (r_1 + r_2)\sigma_1^2} > 1.$$

Hence, we have

$$C_1 = \frac{(r_1\sigma_2^2 - r_1\sigma_1^2 - (r_1 + r_2)\sigma_1^2)\varphi_2(\beta)}{(r_1 + r_2)(r_1\sigma_2^2 - r_2\sigma_1^2)}.$$

For (4b), we first note that $f(\theta_2^{\max}) \neq 0$ since $f(\theta_2^{\max}) = 0$ and $\beta = \theta_2^{\max}$ imply the contradiction that $r_1\sigma_1^2 = 0$. Then, we simply apply (4.3) of Lemma 4.1 to (4.1), and get

$$\lim_{\theta_2 \uparrow \theta_2^{\max}} (\theta_2^{\max} - \theta_2)^{\frac{1}{2}}\psi(\theta_2) = \frac{\varphi_2(\theta_1^{\max})}{f(\theta_2^{\max})}\sqrt{\frac{-\xi_2''(\theta_1^{\max})}{2}} \equiv C_1'.$$

Hence, by (S1) with $s = \frac{1}{2}$, we have (4b), where C_2 is given by

$$C_2 = \frac{C_1'}{\Gamma(\frac{1}{2})} = \frac{\varphi_2(\theta_1^{\max})}{\frac{1}{2}\sigma_2^2\theta_2^{\max} - r_2}\sqrt{\frac{-\xi_2''(\theta_1^{\max})}{2\pi}}.$$

For (4c), we assume that $\frac{1}{2}\alpha_1 < \beta$, which implies $\beta \neq \theta_2^{\max}$. From (4.1),

$$\begin{aligned}\psi(\theta_2) &= \frac{\varphi_2(\xi_1(\theta_2))}{f(\theta_2)(\theta_2 - \theta_1^{\max} + \theta_1^{\max} - \xi_1(\theta_2))} - \frac{\frac{1}{2}\sigma_2^2\theta_2 + r_1}{\theta_2 f(\theta_2)} \\ &= \frac{\varphi_2(\xi_1(\theta_2))(\theta_2 - \theta_1^{\max} - (\theta_1^{\max} - \xi_1(\theta_2)))}{f(\theta_2)((\theta_2 - \theta_1^{\max})^2 - (\theta_1^{\max} - \xi_1(\theta_2))^2)} - \frac{\frac{1}{2}\sigma_2^2\theta_2 + r_1}{\theta_2 f(\theta_2)}. \qquad (4.5)\end{aligned}$$

Since $\theta_2^{\max} - \xi_1(\theta_2^{\max}) > 0$, the denominators in (4.5) do not vanish at $\theta_2 = \theta_2^{\max}$. By the Taylor expansion of $\varphi_2(z)$ at $z = \theta_1^{\max}(= \frac{1}{2}\alpha_1)$,

$$\varphi_2(\xi_1(\theta_2)) = \varphi_2(\theta_1^{\max}) + \varphi_2'(\theta_1^{\max})(\xi_1(\theta_2) - \theta_1^{\max}) + o(|\xi_1(\theta_2) - \theta_1^{\max}|),$$

where we have used the fact that $\varphi_2(z)$ is analytic for $\Re z < \alpha_1$ by Lemma 3.3. Hence, (4.5) can be written as

$$\psi(\theta_2) = (\xi_1(\theta_2) - \theta_1^{\max})K_1(\theta_2) + K_2(\theta_2) + o(|\xi_1(\theta_2) - \theta_1^{\max}|),$$

where $K_1(\theta_2)$ and $K_2(\theta_2)$ are given by

$$\begin{aligned}K_1(\theta_2) &= \frac{\varphi_2(\theta_1^{\max}) + (\theta_2 - \theta_1^{\max})\varphi_2'(\theta_1^{\max})}{f(\theta_2)((\theta_2 - \theta_1^{\max})^2 - (\theta_1^{\max} - \xi_1(\theta_2))^2)} \\ K_2(\theta_2) &= \frac{\varphi_2(\theta_1^{\max})(\theta_2 - \theta_1^{\max})}{f(\theta_2)((\theta_2 - \theta_1^{\max})^2 - (\theta_1^{\max} - \xi_1(\theta_2))^2)} - \frac{\sigma_2^2\theta_2 + 2r_1}{\theta_2(\sigma_2^2\theta_2 - 2r_2)}.\end{aligned}$$

Note that $K_1(\theta_2^{\max})$ is a positive constant since $f(\theta_2^{\max}) = \frac{1}{2}\sigma_2^2\theta_2^{\max} - r_2 > 0$ and $\theta_2^{\max} - \theta_1^{\max} > 0$. Thus, by Lemma 4.1, we have

$$\psi(\theta_2) = -(\theta_2^{\max} - \theta_2)^{\frac{1}{2}} K_1(\theta_2^{\max}) \sqrt{\frac{2}{-\xi_2''(\theta_1^{\max})}} + K_2(\theta_2^{\max}) + o((\theta_2^{\max} - \theta_2)^{\frac{1}{2}}).$$

Hence, (4c) is obtained by (S2), where C_3 is given by

$$\begin{aligned} C_3 &= \frac{1}{\Gamma(-\frac{1}{2})} K_1(\theta_2^{\max}) \sqrt{\frac{2}{-\xi_2''(\theta_1^{\max})}} \\ &= \frac{\varphi_2(\theta_1^{\max}) + (\theta_2^{\max} - \theta_1^{\max})\varphi_2'(\theta_1^{\max})}{(\frac{1}{2}\sigma_2\theta_2^{\max} - r_2)(\theta_2^{\max} - \theta_1^{\max})^2 \sqrt{-2\pi\xi_2''(\theta_1^{\max})}}. \end{aligned}$$

□

In principle, the rough decay rate α_2 is known for a more general two dimensional reflected Brownian queueing network in the framework of large deviations theory. Namely, the rate function for the sample path large deviations is obtained in [2]. We here sharpen the rough decay rate to exact asymptotics. Nevertheless, the rough decay rate in the present form may be also interesting since it clearly explains how the presence of the exogenous input at node 2 decreases α_2.

We next consider exact asymptotics for convex type combination $d_1L_1 + d_2L_2$ with $d_1, d_2 > 0$. Technically, they can be obtained by the same method as Theorem 4.1. However, a new prefactor occurs in asymptotic functions, which corresponds to similar results in Corollary 4.4 of [17]. Furthermore, the exact asymptotics for $L_1 + L_2$ may have its own interest. So, it may be interesting to see what happens in this case. Let

$$\psi_{\boldsymbol{d}}(\theta) = \int_0^\infty e^{\theta x} P(d_1L_1 + d_2L_2 > x)dx.$$

Then,

$$\psi_{\boldsymbol{d}}(\theta) = \frac{1}{\theta}(\varphi(d_1\theta, d_2\theta) - 1),$$

where it follows from (2.11) that

$$\gamma(d_1\theta, d_2\theta)\varphi(d_1\theta, d_2\theta) = (d_1 - d_2)\theta\varphi_1(d_2\theta) + d_2\theta\varphi_2(d_1\theta). \tag{4.6}$$

Hence, we only need to consider the singularity of $\varphi(d_1\theta, d_2\theta)$ similarly to Theorem 4.1. It must occur at the point $\theta(d_1, d_2)$ across the boundary of $\mathcal{D}$, so $d_1\theta \le \alpha_1$ and $d_2\theta \le \alpha_2$. Furthermore, (4.6) and Lemma 3.1 tell that this singularity is caused by the following factors:

(F1) $\gamma(d_1\theta, d_2\theta) = 0$,

(F2) the singularity of $\varphi_2(d_1\theta)$,

(F3) the singularity of $\varphi_1(d_2\theta)$.

For example, the first case occurs when $\theta < \min(\frac{\alpha_1}{d_1}, \frac{\alpha_2}{d_2})$. Note that this singularity is a simple pole since both sides of (4.6) are finite for all θ less than $\min(\frac{\alpha_1}{d_1}, \frac{\alpha_2}{d_2})$. On the other hand, if either $\theta = \frac{\alpha_1}{d_1}$ or $\theta = \frac{\alpha_2}{d_2}$ holds, the singularity is also caused by the corresponding $\varphi_i(d_i\theta)$. Taking these facts into account, we have the following exact asymptotics.

Theorem 4.2 Under the assumptions of Theorem 4.1, for $d_1, d_2 > 0$, let $\delta(d_1, d_2)$ be the non-zero solution θ of $\gamma(d_1\theta, d_2\theta) = 0$. Then, we have

$$\delta(d_1, d_2) = \frac{2(r_1 d_1 + r_2 d_2)}{(\sigma_1^2 d_1^2 + \sigma_2^2 d_2^2)}, \qquad \xi_2(\alpha_1) = \frac{2r_2^+}{\sigma_2^2},$$

and the exact asymptotics of $P(d_1L_1 + d_2L_2 > x)$ has the form of $Ch(x)$ for some constant $C > 0$, where $h(x)$ is given by

(4d) If $\beta < \frac{1}{2}\alpha_1$, then

$$h(x) = \begin{cases} e^{-\frac{\beta}{d_2}x}, & \beta d_1 < (\alpha_1 - \beta)d_2, \\ xe^{-\frac{\beta}{d_2}x}, & \beta d_1 = (\alpha_1 - \beta)d_2, \\ e^{-\delta(d_1,d_2)x} & (\alpha_1 - \beta)d_2 < \beta d_1, \xi_2(\alpha_1)d_1 < \alpha_1 d_2, \\ xe^{-\frac{\alpha_1}{d_1}x} & 0 < \xi_2(\alpha_1)d_1 = \alpha_1 d_2, \\ e^{-\frac{\alpha_1}{d_1}x} & \alpha_1 d_2 < \xi_2(\alpha_1)d_1. \end{cases}$$

(4e) If $\frac{1}{2}\alpha_1 = \beta$, then

$$h(x) = \begin{cases} x^{-\frac{1}{2}}e^{-\frac{\theta_2^{\max}}{d_2}x}, & d_1 \le d_2, \\ e^{-\delta(d_1,d_2)x} & d_1 > d_2, \xi_2(\alpha_1)d_1 < \alpha_1 d_2, \\ xe^{-\frac{\alpha_1}{d_1}x} & 0 < \xi_2(\alpha_1)d_1 = \alpha_1 d_2, \\ e^{-\frac{\alpha_1}{d_1}x} & \alpha_1 d_2 < \xi_2(\alpha_1)d_1. \end{cases}$$

(4f) If $\frac{1}{2}\alpha_1 < \beta < \alpha_1$, then

$$h(x) = \begin{cases} x^{-\frac{3}{2}}e^{-\frac{\theta_2^{\max}}{d_2}x}, & \theta_2^{\max}d_1 < \frac{1}{2}\alpha_1 d_2, \\ x^{-\frac{1}{2}}e^{-\frac{\theta_2^{\max}}{d_2}x}, & \theta_2^{\max}d_1 = \frac{1}{2}\alpha_1 d_2, \\ e^{-\delta(d_1,d_2)x} & \frac{1}{2}\alpha_1 d_2 < \theta_2^{\max}d_1, \xi_2(\alpha_1)d_1 < \alpha_1 d_2, \\ xe^{-\frac{\alpha_1}{d_1}x} & 0 < \xi_2(\alpha_1)d_1 = \alpha_1 d_2, \\ e^{-\frac{\alpha_1}{d_1}x} & \alpha_1 d_2 < \xi_2(\alpha_1)d_1. \end{cases}$$

(4g) If $\alpha_1 \le \beta$, then

$$h(x) = \begin{cases} x^{-\frac{3}{2}}e^{-\frac{\theta_2^{\max}}{d_2}x}, & \theta_2^{\max}d_1 < \frac{1}{2}\alpha_1 d_2, \\ x^{-\frac{1}{2}}e^{-\frac{\theta_2^{\max}}{d_2}x}, & \theta_2^{\max}d_1 = \frac{1}{2}\alpha_1 d_2, \\ e^{-\delta(d_1,d_2)x}, & \frac{\xi_2(\alpha_1)}{\alpha_1} < \frac{d_2}{d_1} < \frac{2\theta_2^{\max}}{\alpha_1}, \\ xe^{-\frac{\alpha_1}{d_1}x} & 0 < \xi_2(\alpha_1)d_1 = \alpha_1 d_2, \\ e^{-\frac{\alpha_1}{d_1}x} & \alpha_1 d_2 < \xi_2(\alpha_1)d_1. \end{cases}$$

Remark 4.2 In (4d), (4e) and (4f), the last two cases vanish if $r_2 \le 0$, equivalently, $\xi_2(\alpha_1) = 0$, but they are always the cases in (4g) since $\alpha_1 \le \beta$ implies $r_2 > 0$. See Figures 1 and 2 for these facts.

Proof. Consider case (4d). From our discussions, we only need to consider the case that $d_1 = d_2$, $\beta d_1 = (\alpha_1 - \beta) d_2$ or $\xi_2(\alpha_1) d_1 = \alpha_1 d_2$ with $r_2 > 0$ holds. If $d_1 = d_2$, then, from (4.6)

$$\gamma(d_1\theta, d_2\theta)\varphi(d_1\theta, d_2\theta) = d_2\theta\varphi_2(d_1\theta).$$

Hence, the singularity of $\varphi(d_1\theta, d_2\theta)$ has a simple pole at $\theta = \frac{\beta}{d_2}$ since (F1) occurs there. Clearly, this case is included in the first case of $h(x)$ in (4d). Consider the case that $\beta d_1 = (\alpha_1 - \beta) d_2$. In this case, the singularity is caused by (F1) and (F3). From this fact and Theorem 4.1, it is not hard to see that, for some positive constant C,

$$\lim_{\theta \uparrow \frac{\theta_2^{\max}}{d_2}} (d_2\theta - \theta_2^{\max})^2 \varphi(d_1\theta, d_2\theta) = C.$$

Hence, (S1) yields the second case of $h(x)$ in (4d). Similarly, we can get the forth case for $\xi_2(\alpha_1) d_1 = \alpha_1 d_2$.

Case (4e) is simpler, and similarly proved. For case (4f), we only consider the case that $\theta_2^{\max} d_1 = \frac{1}{2}\alpha_1 d_2$. In this case, the singularity is again caused by (F1) and (F3). By Theorem 4.1, for some positive constants K', C',

$$\varphi_1(\theta) = K' - C'(\theta_2^{\max} - \theta)^{\frac{1}{2}} + o((\theta_2^{\max} - \theta)^{\frac{1}{2}}), \qquad \theta \uparrow \theta_2^{\max},$$

so (F1) yields, for some positive constant C'',

$$\lim_{\theta \uparrow \frac{\theta_2^{\max}}{d_2}} (d_2\theta - \theta_2^{\max})^{\frac{3}{2}} \varphi(d_1\theta, d_2\theta) = C''.$$

Hence, (S1) implies the second case of $h(x)$ in (4f). We finally consider case (4f). In this case, there are two cases that the singularity is caused by (F1) and either (F2) or (F3), but they can be similarly handled to those in (4f) and (4d). Hence, it is not hard to verify (4g). This completes the proof. □

5 Extensions for the Levy inputs case

We now extend the Brownian tandem queue to the Lévy-driven tandem queue, provided $X_1(t)$ and $X_2(t)$ are independent with positive jumps. In this case, we have to use (3.1) for γ instead of (3.2), but all the arguments can be straightforwardly extended with some extra light-tail conditions. So, we outline the arguments. As mentioned in Example 2.1, we need the following condition for the first queue to have the light-tailed stationary distribution.

(5-i) $c_1\theta = \kappa_1(\theta)$ has a positive solution α_1, and $\kappa_1(\alpha_1 + \epsilon) < \infty$ for some $\epsilon > 0$.

Note that (5-i) implies that $c_1 - \kappa_1'(\theta) = 0$ has a positive solution since $\kappa_1(\theta)$ is convex and non-decreasing, and vanishes at $\theta = 0$. This solution θ is identical with $\theta_1^{\max}$ of Section 3, so we continue to use the same notation. From the arguments on the domain of φ in Section 3, we need a positive solution θ for the equation $\gamma(\theta_1^{\max}, \theta) = 0$ for the second queue to be light-tailed in some cases. So, we assume

(5-ii) $(c_2 - c_1)\theta - \kappa_2(\theta) = \kappa_1(\theta_1^{\max}) - c_1\theta_1^{\max}$ has a positive solution $\theta_2^{\max}$, and $\kappa_2(\theta_2^{\max} + \epsilon) < \infty$ for some $\epsilon > 0$.

This $\theta_2^{\max}$ also corresponds with that of Section 3. Throughout this section, we assume (5-i) and (5-ii) in addition to the stability condition (2-ii). Note that our first requirement (2-i) is automatically satisfied under these two conditions.

We next show how to extend Lemmas 3.1, 3.2 and 3.3. Obviously, Lemmas 3.1 and 3.3 are still valid since the specific form of γ is not used there. To consider Lemma 3.2, recall that r_1 and r_2 are:

$$r_1 = c_1 - \lambda_1, \qquad r_2 = c_2 - c_1 - \lambda_2,$$

and let $\tilde{\kappa}_i(\theta) = \kappa_i(\theta) - \lambda_i\theta$, that is,

$$\tilde{\kappa}_i(\theta) = \frac{1}{2}\sigma_i^2\theta^2 + \kappa_i^{(0)}(\theta) + \kappa_i^{(1)}(\theta) - \lambda_i\theta, \qquad i = 1, 2.$$

Then, γ of (3.1) can be written as

$$\gamma(\theta_1, \theta_2) = r_1\theta_1 + r_2\theta_2 - \tilde{\kappa}_1(\theta_1) - \tilde{\kappa}_2(\theta_2).$$

Since $\tilde{\kappa}_i(0) = \tilde{\kappa}_i'(0) = 0$, $\tilde{\kappa}_i(\theta)$ can play the same role as $\frac{1}{2}\sigma_i^2\theta_i^2$ in (3.2). Thus, we have (3.3), but (3.4) is replaced by

$$\varphi_1(\theta_2) = (\frac{1}{\theta_2}\tilde{\kappa}_2(\theta_2) - r_2)\varphi(0, \theta_2) + r_1 + r_2. \tag{5.1}$$

Hence, Lemma 3.2 is still valid.

We further note that $\gamma(\theta_1, \theta_2)$ is well defined for $\theta_1 \leq \alpha_1, \theta_2 \leq \theta_2^{\max}$. Hence, we can consider the sign of the derivative $\frac{d\theta_2}{d\theta_1}$ as $\theta_1 \uparrow \alpha_1$ when (θ_1, θ_2) moves on the curve $\mathcal{C}$ that is defined by

$$\mathcal{C} = \{(\theta_1, \theta_2) \in \mathbb{R}^2; \gamma(\theta_1, \theta_2) = 0, \theta_1 < \alpha_1, \theta_2 < \theta_2^{\max}\}.$$

On this curve, we obviously have

$$c_1 - \kappa_1'(\theta_1) + (c_2 - c_1 - \kappa_2'(\theta_2))\frac{d\theta_2}{d\theta_1} = 0. \tag{5.2}$$

This implies that $r_1 + r_2 \left.\frac{d\theta_2}{d\theta_1}\right|_{\theta_1=\theta_2=0} = 0$. Hence, by the stability condition (2-ii) and the convexity of $\mathcal{C}$, we can always find a unique positive solution of the equation $\gamma(\theta, \theta) = 0$. Denote this solution by β. Obviously, this β is the natural extension of the one in Sections 3 and 4.

Similarly, the sign of the derivative is not positive at $(\theta_1, \theta_2) = (\alpha_1, 0)$ if and only if

$$(c_1 - \kappa_1'(\alpha_1))r_2 \geq 0 \tag{5.3}$$

since $\kappa_2'(0) = \lambda_2$, where $\kappa_1'(\alpha_1)$ is defined as the left-hand derivative:

$$\kappa_1'(\alpha_1) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}(\kappa_1(\alpha_1) - \kappa_1(\alpha_1 - \epsilon)).$$

The condition (5.3) corresponds with cases (a-) and (b-) in Figures 1 and 2. Cases (a+), (b+) and (c) of those figures occur if and only if (5.3) does not hold. We also define $\eta_i^{\min}$ and $\xi_i(\theta)$ for $i = 1, 2$ in the exactly same way as in Section 3. Then, for $\eta_2^{\min} \leq \theta_2 \leq \theta_2^{\max}$, $\theta_1 = \xi_1(\theta_2)$ if and only if $\theta_2 = \xi_2(\theta_1)$. Furthermore, (3.10) is valid for $0 < \theta_2 < \min(\beta, \alpha_1)$.

We now have all the materials to get the domain of φ, which is also denoted by $\mathcal{D}$, in the same way as in Section 3. Thus, we get

Proposition 5.1 For the Lévy driven tandem queue with an intermediate input, if conditions (2-ii), (5-i) and (5-ii) are satisfied, then the interior of $\mathcal{D}$ is given by (3.9).

We now consider the exact asymptotics of the stationary distribution of L_2. We can again apply the previous arguments, and get similar results to Lemma 4.1 and Theorems 4.1 and 4.2. However, there is one different aspect. That is, we here cannot explicitly compute β, $\theta_2^{\max}$ and α_i for $i = 1, 2$, but they are specified as solutions of the corresponding equations. Thus, Theorem 4.1 is extended in the following form.

Theorem 5.1 For the Lévy driven tandem queue satisfying the conditions (2-ii), (5-i) and (5-ii), let $\theta_1^{\max} = \xi_1(\theta_2^{\max})$, and let β be the unique positive solution of $\gamma(\theta, \theta) = 0$. Then $P(L_1 > x)$ has the following exact asymptotics $h(x)$.

(5a) If $\beta < \theta_1^{\max}$ hold, then $h(x) = C_1 e^{-\beta x}$.

(5b) If $\theta_1^{\max} \leq \beta$ with $\beta = \theta_2^{\max}$ holds, then $h(x) = C_2 x^{-\frac{1}{2}} e^{-\theta_2^{\max} x}$.

(5c) If $\theta_1^{\max} \leq \beta$ with $\beta \neq \theta_2^{\max}$ holds, then $h(x) = C_3 x^{-\frac{3}{2}} e^{-\theta_2^{\max} x}$.

Here, C_i for $i = 1, 2, 3$ are given by

$$C_1 = \frac{\beta \varphi_2(\beta)}{(\xi_1'(\beta) - 1)(\tilde{\kappa}_2(\beta) - r_2\beta)},$$
$$C_2 = \frac{\theta_2^{\max} \varphi_2(\theta_1^{\max})}{\tilde{\kappa}_2(\theta_2^{\max}) - r_2\theta_2^{\max}} \sqrt{\frac{-\xi_2''(\theta_1^{\max})}{2\pi}},$$
$$C_3 = \frac{\theta_2^{\max}(\varphi_2(\theta_1^{\max}) + (\theta_2^{\max} - \theta_1^{\max})\varphi_2'(\theta_1^{\max}))}{(\tilde{\kappa}_2(\theta_2^{\max}) - r_2\theta_2^{\max})(\theta_2^{\max} - \theta_1^{\max})^2 \sqrt{-2\pi\xi_2''(\theta_1^{\max})}},$$

where

$$\xi_1'(\beta) = \frac{\tilde{\kappa}_2'(\beta) - r_2}{r_1 - \tilde{\kappa}_1'(\beta)}, \qquad \xi_2''(\theta_1^{\max}) = \frac{\tilde{\kappa}_1''(\theta_1^{\max})}{r_2 - \tilde{\kappa}_2'(\theta_2^{\max})}.$$

Remark 5.1 From (5.2), it is not hard to see that $\beta < (>)\theta_1^{\max}$ holds if and only if $\left.\frac{d\theta_2}{d\theta_1}\right|_{\theta_1=\beta} > (<)0$ on the curve $\mathcal{C}$, which is equivalent to

$$(c_1 - \kappa_1'(\beta))(c_2 - c_1 - \kappa_2'(\beta)) < (>)0.$$

Similarly, Theorem 4.2 can be extended for the Lévy input case, where the condition $\alpha_1 < \beta$ is replaced by $(c_1 - \kappa_1'(\alpha_1))r_2 > 0$. Since the results are parallel to Theorem 4.2, we omit its details.

We finally note that Theorem 5.1 can be extended to the case that the two components $X_1(t)$ and $X_2(t)$ of Lévy process are not independent but with positive jumps only. In this case, the Lévy exponent $\kappa(\theta_1, \theta_2)$ is defined by

$$E(e^{\theta_1 X_1(t)+\theta_2 X_2(t)}) = e^{t\kappa(\theta_1,\theta_2)}.$$

In this case we have to adapt the Itô's integral formula (2.5), which requires the following extra terms.

$$\int_0^t (f_{12}''(L_1(u), L_2(u))\rho_{12}du + \sum_{0<u\le t} \Delta_{12} f(L_1(u), L_2(u)),$$

where ρ_{12} is the covariance of the Brownian components, and

$$\Delta_{12} f(L_1(u), L_2(u))) = f(L_1(u), L_2(u))) - f(L_1(u-), L_2(u-))).$$

Then, (2.11) still holds, but the γ is changed to

$$\gamma(\theta_1, \theta_2) = c_1\theta_1 + (c_2 - c_1)\theta_2 - \kappa(\theta_1, \theta_2).$$

Although $\kappa_1(\theta_1)$ in (2.13) and $\tilde{\kappa}_2(\theta_2)$ in (5.1) must be changed to $\kappa(\theta_1, 0)$ and $\kappa(0, \theta_2) - \lambda_2\theta_2$, respectively, all the arguments go through under the following conditions corresponding to (5-i) and (5-ii).

(5-ii') $c_1\theta = \kappa(\theta, 0)$ has a positive solution α_1, and $\kappa(\alpha_1 + \epsilon, 0) < \infty$ for some $\epsilon > 0$.

(5-iii') $c_1\theta_1^{\max} + (c_2 - c_1)\theta - \kappa(\theta_1^{\max}, \theta) = 0$ has a positive solution $\theta_2^{\max}$, where $\theta_1^{\max}$ is a positive solution of $\frac{\partial}{\partial\theta_1}\kappa(\theta_1, \theta)|_{\theta_1=\theta_1^{\max}} = c_1$, and $\kappa(\theta_1^{\max}, \theta_2^{\max} + \epsilon) < \infty$ for some $\epsilon > 0$.

6 Concluding remarks

In this section, we first examine the present results to be consistent with existing results. We then discuss possible extensions to other performance characteristics or more general models.

6.1 Compatibility to existing results

Theorem 5.1 generalizes Theorem 4.3 of [13]. Since our notations are different, it is not immediate to see. For the reader convenience we point out relationships between notations; thus $(\mu, \bar{t}, \bar{s}, t_b)$ of [13] $\Rightarrow (\lambda_1, -\beta, -\theta_1^{\max}, -\theta_2^{\max})$. Note that Laplace transforms are used in [13] instead of moment generating functions, so the sign of their variables must be changed. This is the reason why the minus signs appears in the above correspondence. In [13], $t_p \equiv \bar{t}$ and $\theta(s) \equiv c_1 s + \kappa_1(-s)$ are also used, but t_p is always replaced by $\bar{t}$. On the other hand,

$$\theta''(\bar{s}) = \kappa_1''(-\bar{s}) = \kappa_1''(\theta_1^{\max}).$$

Since $\kappa_2(\theta_2) = \tilde{\kappa}_2(\theta_2) = 0$, we have, by Theorem 5.1,

$$\xi''(\theta_2^{\max}) = -\frac{\theta''(\bar{s})}{c_1 - c_2}.$$

Hence, letting $\lambda_2 = 0$ and $\varphi_2(\theta_2) = c_2 - \lambda_1$, we can see that Theorem 5.1 is indeed identical with Theorem 4.3 of [13].

6.2 Rough and exact asymptotics of the joint tail probability

An interesting characteristic, not considered in this paper, is the joint tail distribution $P(L_1 > d_1 x, L_2 > d_2 x)$. Following Proposition 3.2 of [13], an upper bound in the Chernoff inequality

$$\limsup_{x \to \infty} \frac{1}{x} \log P(L_1 > d_1 x, L_2 > d_2 x) \leq -\sup\{d_1\theta_1 + d_2\theta_2; (\theta_1, \theta_2) \in \mathcal{D}^{\mathrm{o}}\}. \tag{6.1}$$

can give the right rough decay rate. Since the set $\mathcal{D}^{\mathrm{o}}$ is explicitly given, it is not hard to find the supremum, which is the maximum over the closure $\overline{\mathcal{D}^{\mathrm{o}}}$. We conjecture that (6.1) is tight, but this seems to be a hard problem. In [13] a sample path large deviation technique was used. This is left for future research.

6.3 The case of more than two nodes

In this subsection, we discuss an extension of our Lévy-driven fluid queue network to n nodes, instead from 2 as it was in previous sections. We are interested in the steady-state behavior of such networks. We hope this will be useful to study the asymptotic tail behavior of their stationary distributions.

Consider n (infinite-buffer) fluid queues, with exogenous input to buffer j in the time interval $[0, t]$ given by $X_j(t)$, where $X_i(t) = a_i t + B_i(t) + J_i(t)$ and

$$\boldsymbol{X}(t) = (X_1(t), \dots, X_n(t))^{\mathrm{T}} = \boldsymbol{a}t + \boldsymbol{B}(t) + \boldsymbol{J}^{(0)}(t) + \boldsymbol{J}^{(1)}(t),$$

where $\boldsymbol{x}^{\mathrm{T}}$ denotes the transpose of column vector $\boldsymbol{x}$. We assume that $\boldsymbol{B}(t)$ is a n-dimensional Brownian motion with null drift, $\boldsymbol{J}^{(0)}(t)$ and $\boldsymbol{J}^{(0)}(t)$ have independent increments which are mutually independent and independent of $\boldsymbol{B}(t)$, $\boldsymbol{J}^{(0)}(t)$ is a martingale which includes all jumps not greater than 1, and $\boldsymbol{J}^{(1)}(t)$ is a pure jump process which is composed of all jumps with sizes greater than 1. We denote the Lévy exponent of $\boldsymbol{X}(t)$ by $\kappa(\boldsymbol{\theta})$.

For a convenience we denote

$$\boldsymbol{Z}(t) = \boldsymbol{B}(t) + \boldsymbol{J}^{(0)}(t) + \boldsymbol{J}^{(1)}(t).$$

The buffers are continuously drained at a constant rate as long as it is not empty. These drain rates are given by a vector $\boldsymbol{c}$; for buffer j, the rate is c_j.

The interaction between the queues is modeled as follows. A fraction p_{ij} of the output of station i is immediately transferred to station j, while a fraction $1 - \sum_{j \neq i} p_{ij}$ leaves the system. We set $p_{ii} = 0$ for all i, and suppose that $\sum_j p_{ij} \leq 1$. The matrix $P = \{p_{ij} : i, j = 1, \ldots, n\}$ is called the *routing matrix*. Assume that

(6-i) $R \equiv I - P^{\mathrm{T}}$ is nonsingular.

We refer to R as reflection matrix.

For $(\{\boldsymbol{X}(t)\}, \boldsymbol{c}, P)$, the *buffer content* process $\boldsymbol{L}(t)$ is defined by

$$\boldsymbol{L}(t) = \boldsymbol{L}(0) + \boldsymbol{Z}(t) + t(\boldsymbol{a} - (I - P^{\mathrm{T}})\boldsymbol{c}) + (I - P^{\mathrm{T}})\boldsymbol{Y}(t),$$

where $\boldsymbol{Y}(t)$ is a *regulator*, that is, the minimal nonnegative and nondecreasing process such that $Y_i(t)$ can be increased only when $L_i(t) = 0$. Denote for a twice continuously differentiable function f of n variables

$$\nabla f = (f'_1, \ldots, f'_n)^{\mathrm{T}}, \qquad \boldsymbol{\Delta} f = (\Delta_1 f, \ldots, \Delta_n f)^{\mathrm{T}}$$

and $\mathcal{L} f = \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2 f''_i$. Using Itô's integral formula (see, e.g., Chapter 26 of [11]) we may write

$$\begin{aligned} f(\boldsymbol{L}(u)) - f(\boldsymbol{L}(0)) \;=\; & \int_0^1 (\nabla f(\boldsymbol{L}(u))^{\mathrm{T}} \, d((\boldsymbol{a} - (I - P^{\mathrm{T}})\boldsymbol{c})u + \boldsymbol{B}(u)) \\ & + \int_0^1 (\nabla f(\boldsymbol{L}(u))^{\mathrm{T}} \, d\boldsymbol{Y}(t) + \int_0^1 \mathcal{L} f(\boldsymbol{L}(u)) \, du \\ & + \sum_{0 \leq u \leq t} (\boldsymbol{\Delta} f(\boldsymbol{L}(u)) 1(\Delta \boldsymbol{J}^{(0)}(u) \neq \boldsymbol{0}) - \Delta \boldsymbol{J}^{(0)}(u) \nabla f(\boldsymbol{L}(u))) \\ & + \sum_{0 \leq u \leq t} \boldsymbol{\Delta} f(\boldsymbol{L}(u)) 1(\Delta \boldsymbol{J}^{(1)}(u) \neq \boldsymbol{0}). \end{aligned} \tag{6.2}$$

Suppose now, for $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)^{\mathrm{T}}$ and $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ we take $f(\boldsymbol{x}) = \exp(\langle \boldsymbol{\theta}, \boldsymbol{x} \rangle)$, where $\langle \boldsymbol{\theta}, \boldsymbol{x} \rangle$ is the inner product of vectors $\boldsymbol{\theta}$ and $\boldsymbol{x}$. Assume now

(6-ii) $\boldsymbol{L}(t)$ has a stationary distribution.

Denote this stationary distribution by π.

For taking the expectation of (6.2) under π, we need the finiteness of $E_\pi(Y_i(1))$ for all $i = 1, 2, \ldots, n$. To verify this, we define n-dimensional process $\boldsymbol{U}(t)$ as

$$\boldsymbol{U}(t) = (I - P^{\mathrm{T}})^{-1} \boldsymbol{L}(t).$$

Then,

$$\boldsymbol{U}(t) = \boldsymbol{U}(0) + (I - P^{\mathrm{T}})^{-1}(t\boldsymbol{a} + \boldsymbol{Z}(t)) - t\boldsymbol{c} + \boldsymbol{Y}(t).$$

By the assumption (6-ii), $\boldsymbol{U}(t)$ has the stationary distribution. Hence, we must have

$$(I - P^{\mathrm{T}})^{-1}(\boldsymbol{a} + E(\boldsymbol{Z}(1))) < \boldsymbol{c}. \tag{6.3}$$

Intuitively, this condition is also sufficient for (6-ii), but we have not yet proved it. So, we keep the assumption (6-ii).

We apply Itô's integral formula to this process $\boldsymbol{U}(t)$, and take the expectation under π. Obviously $\boldsymbol{U}(t)$ is also stationary under π. Then, similarly to our arguments in Section 2, we get the following lemma.

Lemma 6.1 Under conditions (6-i) and (6-ii), $\boldsymbol{L}(t)$ has the stationary distribution π and,

$$E_\pi(\boldsymbol{Y}(1)) = \boldsymbol{c} - (I - P^{\mathrm{T}})^{-1}(\boldsymbol{a} + E(\boldsymbol{Z}(1)))$$

is a finite and positive vector.

Thus, $E_\pi(Y_i(1))$ must be finite. Let $\varphi(\boldsymbol{\theta})$ be the moment generating function of π. Similarly, let

$$\varphi_i(\boldsymbol{\theta}_i[0]) = E_\pi \int_0^1 e^{\langle \boldsymbol{\theta}_i[0], \boldsymbol{L}(u)\rangle}\, dY_i(u),$$

where $\boldsymbol{\theta}_i[0]$ be the n-dimensional vector obtained from $\boldsymbol{\theta}$ by replacing θ_i by 0. Denote the column vector whose i-th entry is $\varphi_i(\boldsymbol{\theta}_i[0])$, that is,

$$\underline{\boldsymbol{\varphi}}(\boldsymbol{\theta}) = (\varphi_1(\boldsymbol{\theta}_1[0]), \ldots, \varphi_n(\boldsymbol{\theta}_n[0]))^{\mathrm{T}}$$

Similarly to the two dimensional case, let

$$\gamma(\boldsymbol{\theta}) = \langle (I - P^{\mathrm{T}})\boldsymbol{c}, \boldsymbol{\theta}\rangle - \kappa(\boldsymbol{\theta}).$$

We are now ready to extend Proposition 2.1 for the general n node case as follows.

Proposition 6.1 Under conditions (6-i) and (6-ii), we have, for $\boldsymbol{\theta} \in \mathbb{R}^n$,

$$\gamma(\boldsymbol{\theta})\varphi(\boldsymbol{\theta}) = \boldsymbol{\theta}^{\mathrm{T}}(I - P^{\mathrm{T}})\underline{\boldsymbol{\varphi}}(\boldsymbol{\theta}), \tag{6.4}$$

as long as $\varphi(\boldsymbol{\theta})$, $\gamma(\boldsymbol{\theta})$ and $\underline{\boldsymbol{\varphi}}(\boldsymbol{\theta})$ are finite, however at least for $\boldsymbol{\theta} \le \boldsymbol{0}$.

PROOF. Similarly to Proposition 2.1, we first take the expectation of (6.2) for twice differentiable and bounded function f, then we get (6.4) by approximating $e^{\langle \boldsymbol{\theta}, \boldsymbol{x}\rangle}$ by a suitably chosen set of f's. □

Similarly to Example 2.1, the explicit solution for φ can be obtained from Proposition 6.1 for Lévy-driven tandem without intermediate inputs, that is when $X_i(t) = 0$ for $i = 2, \ldots$ and $p_{i,i+1} = 1$ and otherwise 0. In what follows, we consider this special case.

We first consider the stability condition (6.3). Since $(P^{\mathrm{T}})^2 = \mathbf{0}$, we have $(I - P^{\mathrm{T}})^{-1} = I + P^{\mathrm{T}}$. Thus we have

$$\text{the } ij\text{-entry of } (I - P^{\mathrm{T}})^{-1} = \begin{cases} 1, & i = j \text{ or } i = j+1 \\ 0, & \text{otherwise.} \end{cases}$$

Therefore the stability condition (6.3) is read:

$$\sum_{i=1}^{j} \lambda_i < c_j, \qquad j = 1, 2, \dots, n, \tag{6.5}$$

where $\lambda_i = a_i + E(Z_i(1))$.

Without loss of generality we may assume that $c_1 > c_2 > \dots > c_n$. For such the network we have the following property that if $L_i(u) = 0$, then $L_j(u) = 0$ for $j = 1, \dots, i$. Let

$$\varphi_i(\boldsymbol{\theta}_i[0]) = \varphi_i(\boldsymbol{\theta}^{i+1}), \qquad i = 1, \dots, n, \tag{6.6}$$

where $\boldsymbol{\theta}^i$ is the n-dimensional vector which replaces the first $i-1$ entries of $\boldsymbol{\theta}$ by zero.

Let for $i = 2, 3, \dots, n$, $[\theta]^{i-1}$ be the $i-1$-dimensional vector whose entries are all θ, and ξ_{i-1} be the minimal solution (provided it exists) of

$$\gamma(([\xi_{i-1}]^{i-1}, \theta_i, \theta_{i+1}, \dots, \theta_n)) = 0,$$

Solution $\xi_{i-1} = \xi_{i-1}(\boldsymbol{\theta}^i)$ is a function of $\boldsymbol{\theta}^i = (0, \dots, 0, \theta_i, \theta_{i+1}, \dots, \theta_n)$. Notice that ξ_j is the minimal solution of

$$c_j \xi_j - \kappa_1(\xi_j) = \sum_{i=j+1}^{n} (c_{i-1} - c_i)\theta_i.$$

Let

$$\zeta_i(\boldsymbol{\theta}^i) = \frac{\theta_i - \xi_i(\boldsymbol{\theta}^{i+1})}{\theta_i - \xi_{i-1}(\boldsymbol{\theta}^i)}, \qquad i = 2, 3, \dots, n-1.$$

We now state result, which is a special case of Theorem 6.1 from [6], with an alternative proof.

Theorem 6.1 For the n-stage tandem queue without intermediate input satisfying the stability condition (6.5), assume that $\xi_{i-1}(\boldsymbol{\theta}^i)$ exists for $i = 2, 3, \dots n$. Then we have, except for finitely many singular points,

$$\varphi(\boldsymbol{\theta}) = \frac{(c_n - \lambda_1)\theta_n}{\gamma(\boldsymbol{\theta})} \left(\sum_{i=1}^{n-1} \frac{\theta_i - \theta_{i+1}}{\theta_n - \xi_{n-1}(\boldsymbol{\theta}^n)} \prod_{j=i+1}^{n-1} \zeta_j(\boldsymbol{\theta}^j) + 1 \right), \qquad \boldsymbol{\theta} \le \mathbf{0}, \tag{6.7}$$

where

$$\gamma(\boldsymbol{\theta}) = c_1\theta_1 + \sum_{i=2}^{n} (c_i - c_{i-1})\theta_i - \kappa_1(\theta_1).$$

Proof. Using (6.6), we can write the stationary equation (6.4) of Proposition 6.1 as

$$\gamma(\boldsymbol{\theta})\varphi(\boldsymbol{\theta}) = \sum_{i=1}^{n-1}(\theta_i - \theta_{i+1})\varphi_i(\boldsymbol{\theta}^{i+1}) + \theta_n \varphi_n(\mathbf{0}). \tag{6.8}$$

By Lemma 6.1, we have $\varphi_i(\mathbf{0}) = c_i - \lambda_i$. Hence, (6.7) is obtained from (6.8) if

$$\varphi_i(\boldsymbol{\theta}^{i+1}) = \frac{(c_n - \lambda_1)\theta_n}{\theta_n - \xi_{n-1}(\boldsymbol{\theta}^n)} \prod_{j=i+1}^{n-1} \zeta_j(\boldsymbol{\theta}^j), \qquad i = 1, 2, \ldots, n-1, \tag{6.9}$$

hold. We inductively derive (6.9) from (6.8). For this, for each $i = 2, 3, \ldots, n$, substitute

$$\boldsymbol{\theta} = ([\xi_{i-1}(\boldsymbol{\theta}^i)]^{i-1}, \theta_i, \theta_{i+1}, \ldots, \theta_n)$$

into (6.8), then we have, for $i = 2, 3, \ldots, n$,

$$(\xi_{i-1}(\boldsymbol{\theta}^i) - \theta_i)\varphi_{i-1}(\boldsymbol{\theta}^i) + \sum_{j=i}^{n-1}(\theta_j - \theta_{j+1})\varphi_j(\boldsymbol{\theta}^{j+1}) + \theta_n \varphi_n(\mathbf{0}) = 0.$$

Taking the difference of the above equations for i and $i+1$, we have, for $i = 2, 3, \ldots, n-1$,

$$(\xi_{i-1}(\boldsymbol{\theta}^i) - \theta_i)\varphi_{i-1}(\boldsymbol{\theta}^i) - (\xi_i(\boldsymbol{\theta}^{i+1}) - \theta_{i+1})\varphi_i(\boldsymbol{\theta}^{i+1}) + (\theta_i - \theta_{i+1})\varphi_i(\boldsymbol{\theta}^{i+1}) = 0,$$

which implies

$$\varphi_{i-1}(\boldsymbol{\theta}^i) = \frac{\theta_i - \xi_i(\boldsymbol{\theta}^{i+1})}{\theta_i - \xi_{i-1}(\boldsymbol{\theta}^i)} \varphi_i(\boldsymbol{\theta}^{i+1}) = \zeta_i(\boldsymbol{\theta}^i)\varphi_i(\boldsymbol{\theta}^{i+1}), \qquad i = 2, 3, \ldots, n-1,$$

and

$$(\xi_{n-1}(\boldsymbol{\theta}^n) - \theta_n)\varphi_{n-1}(\boldsymbol{\theta}^n) + \theta_n \varphi_n(\mathbf{0}) = 0.$$

Hence, we have (6.9) since $\varphi_n(\mathbf{0}) = c_n - \lambda_1$. This completes the proof. □.

It may be interesting to study the asymptotic behavior of the marginal distribution of L_i using Theorem 6.1. As we already noted, this has been done for $n = 2$ in [13]. For general $n \geq 3$, (6.7) may be still useful. However, our approach to work directly on (6.8) can be also fruitful here. We leave this for future research.

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A Computation of $\xi_1'(\beta)$

From the definition of $\xi_1(\theta)$, we have

$$\xi_1(\theta) = \frac{1}{\sigma_1^2}\left(r_1 - \sqrt{r_1^2 - \sigma_1^2(\sigma_2^2\theta^2 - 2r_2\theta)}\right),$$

which implies

$$\xi_1'(\theta) = \frac{\sigma_2^2\theta - r_2}{\sqrt{r_1^2 - \sigma_1^2(\sigma_2^2\theta^2 - 2r_2\theta)}},$$

as long as $r_1^2 - \sigma_1^2(\sigma_2^2\theta^2 - 2r_2\theta) > 0$. This is always the case if $\theta \le \beta < \frac{1}{2}$. From $\gamma(\beta, \beta) = 0$, we have $2r_1 - \sigma_1^2\beta = \sigma_2^2\beta - 2r_2$. Hence,

$$r_1^2 - \sigma_1^2(\sigma_2^2\beta^2 - 2r_2\beta) = r_1^2 - \sigma_1^2(2r_1\beta - \sigma_1^2\beta^2) = (r_1 - \sigma_1^2\beta)^2.$$

This implies that

$$\lim_{\theta \to \beta} \xi_1'(\theta) = \frac{\sigma_2^2\beta - r_2}{r_1 - \sigma_1^2\beta}. \tag{A.1}$$