Tail asymptotics of the stationary distribution for a two-node generalized Jackson network

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ABSTRACT

A primary interest of this paper is to find tail asymptotics of the stationary distribution of a generalized Jackson network with two nodes under phase-type setting, provided its stability holds. Here, the phase type setting is meant that the arrival processes are the so called Markov arrival processes, and the service time distributions are of phase type. We consider two types of the tail asymptotics, the tail decay rate of the marginal stationary distribution in an arbitrary direction and those for the joint stationary probabilities in the coordinate directions.

There are two major reasons why those tail asymptotics are interesting particularly for the two-node generalized Jackson network. Before discussing them, we recall what is the generalized Jackson network and how it has been studied.

A queueing network in which at each node customers finishing service are independently routed according to given probabilities and their service times are *i.i.d* is called a generalized Jackson network when exogenous arrivals and service times at each node are independent but their distributions are general. Here, service discipline at each node is assumed to be first-come and first-served. If the exogenous arrival processes are time-homogeneous Poisson and the service times are exponentially distributed, then this network becomes the well known Jackson network. Thus, the generalized Jackson network is a natural generalization of the Jackson network. In this paper, we assume that each node has a single server.

The Jackson network has been widely used in applications because it has a product form stationary distribution, which is analytically tractable. However, this is not the case for the generalized Jackson network. This fact motivated Harrison and Williams [12] to study diffusion approximation for such networks in heavy traffic. They derived the prototype of a semimartingale reflecting Brownian motion on a nonnegative orthant, SRBM for short. Since then SRBM has been extensively studied, which became a big research area (see [5]). Furthermore, its relation to the generalized Jackson network is still a hot topic (e.g., see [3, 10]).

We now return to the reasons for our study. When a stationary distribution is difficult to analyze, it is natural to study its characteristics which are tractable in analysis but still important in application. Tail asymptotic is one of such characteristics, and has been actively studied for many years (see, e.g., [16] and references therein). However, we still do not have a satisfactory answer even for the two-node generalized Jackson network with single servers. This motivates us to study its tail asymptotic problem.

Another reason is related to the two-dimensional SRBM which is obtained as a diffusion limit of a sequence of the two-node generalized Jackson networks (see [10] for its verification concerning stationary distribution). We now have good view on the tail asymptotics of the stationary distribution of a two-dimensional SRBM (e.g., see [1, 6, 7]). Hence, if we can get those for the two-node generalized Jackson network, then we can argue the quality of the diffusion approximation by them. In this consideration as well as applications of the generalized Jackson network itself, it is desirable to get the tail asymptotics in analytically tractable form.

To facilitate our analysis, we assume the phase-type setting. This certainly limits the class of queueing network models, but we believe it is sufficiently large since phase-type distributions are dense in the set of distributions on the nonnegative real line, and Markovian arrival processes are versatile. For example, the exogenous arrival processes are not necessarily to be renewal processes, which are often assumed for the generalized Jackson network (e.g., see [10]).

We consider the two-node generalized Jackson network under phase-type setting as a special case of a Markov modulated two-dimensional reflecting random walk, and study the tail asymptotic problem for this reflecting process. Thus, we answer the problem for a more general model.

Recently, Ozawa [18] introduced a two-dimensional quasi-birth-and-death process, 2d-QBD process for short, for queueing networks, and studied the tail asymptotics of its stationary distribution. We will take this 2d-QBD process for considering the tail asymptotic problem. However, we can not use Ozawa's [18] results because they require to numerically solve certain parametrized matrix equations over some range of parameters. Thus, the tail asymptotics obtained in [18] are analytically intractable. Furthermore, the asymptotics are only considered for the coordinate directions.

Thus, we need to answer the problem in a different way from Ozawa's [18]. To this end, we can see that a crucial step is to find the convergence parameter of a nonnegative infinite dimensional matrix with QBD like block matrices. This problem can be reduced to the existence of a nonnegative right invariant vector of such a matrix, and we refer to this vector as a super-harmonic vector. Thus, the problem is to find a necessary and sufficient condition in terms of block matrices for the existence of the super-harmonic vector. However, those obtained in [18] are not analytically

tractable as already mentioned. So, we make an assumption to have a tractable necessary and sufficient condition, which must be easily checkable. We make similar assumptions for the 2d-QBD process, and verify that they are satisfied by the two-node generalized Jackson network. Details for those arguments can be found in [17].

Once this convergence parameter problem is resolved, we can extend all the techniques recently used to study a two-dimensional reflecting random walk with unbounded jumps in [15] for that under Markov modulation. In this way, we obtain the tail asymptotics of the stationary distribution of the two-node generalized Jackson network. Namely, let $\boldsymbol{Z} \equiv (L_1, L_2, J)$ be a random vector subject to the stationary distribution of the numbers of customers L_1 and L_2 at two nodes and the background state J describing the phases of arrival streams and progress of service. Then, for any two dimensional non-zero vector $\boldsymbol{c} \geq \boldsymbol{0}$, the following limit exists and is obtained in terms of the modeling parameters.

$$\frac{1}{x} \log \mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L} \rangle > x) \qquad (x \to \infty), \tag{1}$$

where $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ is the standard inner product of vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$. We refers to this decay rate as that of the marginal stationary distribution in diction $\boldsymbol{c}$. Similarly, we obtain the following limit for each fixed ℓ and k and $i = 1, 2$.

$$\frac{1}{n} \log \mathbb{P}(L_i = n, L_{3-i} = \ell, J = k) \qquad (n \to \infty). \tag{2}$$

Those results are given in Theorem 2 below.

The tail asymptotic problem for the two-node generalized Jackson network under phase-type setting has been studied by Yukio Takahashi and his colleagues [13, 14], but they only derive upper bounds for the tail decay rates of the stationary probabilities. There are many studies for the tandem queue case (e.g., see [2, 4, 11, 9, 19]). However, those studies are also incomplete because either extra conditions are required or the tail asymptotics are only obtained for the marginal stationary distribution for one node. By contrast, we derive the tail decay rate for the marginal distribution in an arbitrary direction (see (1)), and do not require any extra condition except for the phase-type setting.

In what follows, we summarize our main results. We first formally introduce the two-node Jackson network under phase-type setting. We number the two node as 1 and 2, and assume the following dynamics.

(a) A customer which completes service at node i independently goes to node j with probability r_{ij} for $i = 1, 2, j = 0, 1, 2$, where node 0 means the outside of the network, $r_{12} + r_{21} > 0$ and $r_{12} r_{21} < 1$. We assume that $r_{ii} = 0$ for simplicity.

(b) Exogenous customers arrive at node i subject to the Markovian arrival process with generator $T_i + U_i$ and generating arrivals by rate matrix U_i. Here, T_i and U_i are finite square matrices of the same size for each $i = 1, 2$. Let $\boldsymbol{\nu}_i$ be the stationary distribution of $T_i + U_i$, then the mean exogenous arrival rate at node i is given by $\lambda_i \equiv \langle \boldsymbol{\nu}_i, U_i \boldsymbol{1} \rangle$, where $\boldsymbol{1}$ is the column vector whose entries are all units.

(c) Node i has a single server, whose service times are independently and identically distributed subject to a phase type distribution with $(\boldsymbol{\beta}_i, S_i)$, where $\boldsymbol{\beta}_i$ is the row vector representing the initial phase distribution and S_i is a transition rate matrix for internal state transitions. Here, S_i is a finite square matrix, and $\boldsymbol{\beta}_i$ has the same dimension as that of S_i for each $i = 1, 2$. The mean service rate at node i is given by $\mu_i = \langle \boldsymbol{\beta}_i, (-S_i)^{-1} \boldsymbol{1} \rangle$.

It is well known that the generalized Jackson network has the stationary distribution if and only if

$$\rho_i \equiv \frac{\lambda_i + \lambda_{(3-i)} r_{(3-i)i}}{\mu_i (1 - r_{12} r_{21})} < 1, \qquad i = 1, 2. \tag{3}$$

We assume this condition throughout the paper.

For $i = 1, 2$ and $\boldsymbol{\theta} = (\theta_1, \theta_2)^{\mathrm{T}}$, let

$$t_i(\boldsymbol{\theta}) = e^{-\theta_i} (r_{i0} + e^{\theta_{3-i}} r_{i(3-i)}),$$

where superscript $^{\mathrm{T}}$ stands for transpose, and let $\gamma^{(ia)}(\theta_i)$ and $\gamma^{(id)}(\boldsymbol{\theta})$ be the Perron-Frobenius eigenvalues of $T_i + e^{\theta_i} U_i$ and $S_i + t_i(\boldsymbol{\theta}) D_i$, respectively, where $D_i = (-S_i \boldsymbol{1}) \boldsymbol{\beta}_i$. That is, there are positive column vectors $\boldsymbol{h}^{(ia)}(\theta_i)$, $\boldsymbol{h}^{(id)}(\boldsymbol{\theta})$ such that

$$\begin{aligned} (T_i + e^{\theta_i} U_i) \boldsymbol{h}^{(ia)}(\theta_i) &= \gamma^{(ia)}(\theta_i) \boldsymbol{h}^{(ia)}(\theta_i), \\ (S_i + t_i(\boldsymbol{\theta}) D_i) \boldsymbol{h}^{(id)}(\boldsymbol{\theta}) &= \gamma^{(id)}(\boldsymbol{\theta}) \boldsymbol{h}^{(id)}(\boldsymbol{\theta}), \quad i = 1, 2. \end{aligned}$$

Let g_i be the moment generating function of the service time distribution at node i, then it can be shown that

$$\gamma^{(id)}(\boldsymbol{\theta}) = -g_i^{-1}((t_i(\boldsymbol{\theta}))^{-1}), \qquad i = 1, 2. \tag{4}$$

Those Perron-Frobenius eigenvalues also have the following interpretations. Let $N_{ia}(t)$ be the number of arrivals at node i during the time period $[0, t]$, then

$$\gamma^{(ia)}(\theta_i) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(e^{\theta_i N_{ia}(t)}). \tag{5}$$

Similarly, let $N_{id}(t)$ be the number of departures from node i during the time period $[0, t]$ and let $\Phi_{ij}(n)$ is the number of customers who are routed to node j among n departing customers at node i, then

$$\mathbb{E}(t_i(\boldsymbol{\theta})^{N_{id}(t)}) = \mathbb{E}(e^{-\theta_i N_{id}(t) + \theta_{3-i} \Phi_{i(3-i)}(N_{id}(t))}),$$

and therefore

$$\gamma^{(id)}(\boldsymbol{\theta}) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}(e^{-\theta_i N_{id}(t) + \theta_{3-i} \Phi_{i(3-i)}(N_{id}(t))}). \tag{6}$$

Thus, $\gamma^{(ia)}(\theta_i)$ and $\gamma^{(id)}(\boldsymbol{\theta})$ are cumulant generating functions in the theory of large deviations (e.g., see [8]).

We are now ready to present main results. For them, the following notations are convenient.

$$\begin{aligned} \partial\Gamma &= \{\boldsymbol{\theta} \in \mathbb{R}^2; \gamma^{(1a)}(\theta_1) + \gamma^{(2a)}(\theta_2) \\ &\qquad + \gamma^{(1d)}(\boldsymbol{\theta}) + \gamma^{(2d)}(\boldsymbol{\theta}) = 0\}, \\ \Gamma_{\max} &= \{\boldsymbol{\theta} \in \mathbb{R}^2; \exists \boldsymbol{\theta}' \in \partial\Gamma, \boldsymbol{\theta} < \boldsymbol{\theta}'\}, \\ \partial\Gamma_i &= \{\boldsymbol{\theta} \in \partial\Gamma; t_i(\boldsymbol{\theta}) \geq 1\}, \qquad i = 1, 2. \end{aligned}$$

Remark 1. *If we define* Γ *as*

$$\begin{aligned} \Gamma = \{\boldsymbol{\theta} \in \mathbb{R}^2; &\gamma^{(1a)}(\theta_1) + \gamma^{(2a)}(\theta_2) \\ &+ \gamma^{(1d)}(\boldsymbol{\theta}) + \gamma^{(2d)}(\boldsymbol{\theta}) \leq 0\}, \end{aligned}$$

then it can be shown that Γ *is a convex set. Hence,* $\partial\Gamma$ *can be considered as a convex curve.*

Let $\varphi(\boldsymbol{\theta}) \equiv \mathbb{E}(e^{\langle \boldsymbol{\theta}, \boldsymbol{L} \rangle})$ be the moment generating function of $\boldsymbol{L}$ subject to the marginal stationary distribution with respect to the numbers of customers at nodes 1 and 2, and define $\mathcal{D}$ as

$$\mathcal{D} = \text{the interior of } \{\boldsymbol{\theta} \in \mathbb{R}^2; \varphi(\boldsymbol{\theta}) < \infty\},$$

which is referred to as a convergence domain of φ. Denote the two extreme points of Γ_i by

$$\boldsymbol{\theta}^{(i,\Gamma)} = \arg_{\boldsymbol{\theta} \in \mathbb{R}^2} \sup\{\theta_i \geq 0; \boldsymbol{\theta} \in \partial \Gamma_i\}, \quad i = 1, 2.$$

Using these points, we define the vector $\boldsymbol{\tau}$ by

$$\tau_1 = \sup\{\theta_1 \in \mathbb{R}; \boldsymbol{\theta} \in \partial \Gamma_1; \theta_2 < \theta_2^{(2,\Gamma)}\},$$
$$\tau_2 = \sup\{\theta_2 \in \mathbb{R}; \boldsymbol{\theta} \in \partial \Gamma_2; \theta_1 < \theta_1^{(1,\Gamma)}\}.$$

We have the following two theorems (see [17] for their proofs).

Theorem 1. *For the two node generalized Jackson network under phase type setting, if it is stable, then*

$$\mathcal{D} = \{\boldsymbol{\theta} \in \Gamma_{\max}; \boldsymbol{\theta} < \boldsymbol{\tau}\}. \tag{7}$$

Theorem 2. *Under the same assumptions of Theorem 1,*

$$\lim_{x \to \infty} \frac{1}{x} \log \mathbb{P}(\langle \boldsymbol{c}, \boldsymbol{L} \rangle > x) = -\sup\{u > 0; u\boldsymbol{c} \in \mathcal{D}\}, \tag{8}$$

and, for each fixed nonnegative integer ℓ and $j \in \mathcal{V}^{(i)}_{\max(1,\ell)}$,

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(L_i = n, L_{3-i} = \ell, J = j)$$
$$= -\sup\{\theta_i > 0; \boldsymbol{\theta} \in \mathcal{D}\} \; (= -\tau_i), \tag{9}$$

where $\mathcal{V}^{(i)}_0$ ($\mathcal{V}^{(i)}_1$) is the set of background states for $L_i \geq 1$ and $L_{3-i} = 0$ ($L_{3-i} \geq 1$, respectively).

Note that these theorems show that the decay rates are determined by (5) and (6). Hence, we can expect that Theorems 1 and 2 hold beyond the phase-type setting and more general routing mechanism.

It is also notable that these decay rates are obtained in the exactly same way as those of the stationary distribution of the two-dimensional SRBM (see Theorems 2.1, 2.2 and 2.3 of [6]). Furthermore, we can prove that the tail decay rates for the two-node generalized Jackson network under diffusion scaling converge to those of the corresponding SRBM. Hence, the decay rates are continuous under the diffusion approximation.

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