Product-form characterization for a two-dimensional reflecting random walk

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September 30, 2013

Abstract

We consider the two dimensional skip free reflecting random walk on the nonnegative integers, which is referred to as a 2-d reflecting random walk. We give necessary and sufficient conditions for the stationary distribution to have a product-form. We also derive simpler sufficient conditions for product-form for a restricted class of 2-d reflecting random walks.

We apply these results and obtain a product-form approximation of the stationary distribution through a suitable modification of the parameters of the random walk.

Keywords: Two dimensional reflecting random walk, stationary distribution, decay rate, product-form, geometric interpretation, approximation.
AMS Classification: 60J10, 82B41.

1 Introduction

We are concerned with a two dimensional reflecting skip-free random walk on the nonnegative integers, which we assume to form an irreducible, positive recurrent, discrete-time Markov chain. Such processes are called a double QBD process in Miyazawa [17]; here, we refer to them as 2-d reflecting random walks or "reflecting random walks" for short.

The discrete-time version of queueing networks can be considered as reflecting random walks in many cases, and their stationary distributions are known to have product-form under certain conditions such as local balance (see Chao *et al.* [3] and Serfozo [22] for instance). The two-node tandem queue in discrete-time, in particular, is a special case of 2-d random walk which has a product-form stationary distribution (Hsu and Burke [12]). These cases are, however, rather exceptional even for 2-d reflecting random walks, and except for them, it is generally hard to determine the stationary distribution in a closed form, so that recent interest has focused on the tail asymptotic properties of the stationary distribution. These have been well studied recently, see Miyazawa [18] and references therein.

We proceed in another direction in the present paper. We are interested in characterizing a class of 2-d reflecting random walks, the stationary distributions of which have product-form. Our motivation is double. Firstly, product-form distributions are analytically highly tractable and enable one to easily compute various performance characteristics. We want to know under which conditions product-form stationary distributions arise in 2-d reflecting random walks. This is of theoretical interest of course but, as we shall see, such knowledge is useful to find product-form approximations for otherwise unmanageable random walks.

We are also motivated by the recent work of Latouche, Mahmoodi and Taylor [15], where it is shown that one may force the stationary distribution for $GI/M/1$-type processes to have product-form by modifying the transition probabilities at level zero. It is important to note that, in [15] as well as here, conditions for product-form are expressed in terms of the modeling primitives, that is, in terms of the characterization one gives to the transition probabilities.

Our necessary and sufficient condition for product-form is formulated as a system of linear and quadratic equations in terms of the modeling primitives and we give geometrical interpretations to those conditions. For this, we slightly extend the graph representation developed in Miyazawa [17] and draw figures to represent the conditions. It is both intuitively appealing and helpful to see how the modeling primitives influence the product-form distribution. Further details on this geometrical view are to be found in Miyazawa [18], a recent survey paper.

Conditions for product-form stationary distributions have been formulated in Chao *et al.* [4]; some of the queueing models considered there may be formulated as multidimensional reflecting random walks. However, those characterizations generally use the stationary distribution which is unknown. A similar type of results is to be found in Bayer and Boucherie [1]. In the present paper, we consider 2-d reflecting random walks, which are much simpler than those multidimensional random walks, but our characterization specifically gives a way to find the modeling primitives to have the product-form stationary distribution.

The two-dimensional semi-martingale Brownian motion on the quadrant has recently been analysed in Dai and Miyazawa [5] in the same spirit as here, and that characterization also has a geometric interpretation. The conditions obtained here for the 2-d reflecting random walk are similar but more involved, and we need more constraints to have product-form. We discuss this more precisely in Remark 3.8.

The paper is made up of four sections. We formally introduce in the next section the 2-d reflecting random walk under two formulations, as a random walk and as a quasi-birth-and-death process (QBD for short). We give in Section 3 necessary and sufficient conditions for the stationary distribution to have product-form, and we derive more tractable conditions for a special family. Those conditions have nice geometric interpretations. In Section 4, we illustrate our results and show how they may be used to approximate the stationary distribution of a 2-d reflecting random walk by one with product form. Two approximations are numerically compared, one which minimizes the modification to the modeling primitives while keeping the exact decay rate and one without this constraint.

2 Reflecting random walks

The 2-dimensional reflecting random walk is analogous to a birth-and-death process with homogeneous state transitions. The two coordinates are correlated and the reflection mechanisms on the boundary of the quadrant are arbitrary for each boundary face. We define this process in two ways, both of which are useful for the sequel.

2.1 Modeling primitives

Following [18], we now formally define a 2-d reflecting random walk on the integer quadrant $S \equiv \mathbb{Z}_+^2$, $\mathbb{Z}_+$ being the set of nonnegative integers. We partition the state space S into four subsets:

$$\begin{aligned} S_+ &= \{(i,j) \in S; i,j \geq 1\}, & S_0 &= \{(0,0)\}, \\ S_1 &= \{(i,j) \in S; i \geq 1, j = 0\}, & S_2 &= \{(i,j) \in S; i = 0, j \geq 1\}. \end{aligned}$$

The set S_+ is called the interior and $\partial S \equiv \cup_{k=0,1,2} S_k$ is called the boundary, while the sets S_k themselves, for $k = 0, 1, 2$, are called boundary faces. Consider a discrete time Markov chain $\{\boldsymbol{Z}_\ell\}$ with state space S, and homogeneous transitions on the interior and on each boundary face. We assume that this Markov chain is skip free, that is, $|Z_{(\ell+1)i} - Z_{\ell i}| \leq 1$ for $i = 1, 2$ with probability one, and we denote by $\{p_{ij}^{(k)}; i, j = 0, \pm 1\}$, the distribution of $\boldsymbol{Z}_{\ell+1} - \boldsymbol{Z}_\ell$ given that $\boldsymbol{Z}_\ell \in S_k$, for $k = +, 0, 1, 2$. Finally, we assume that $\{\boldsymbol{Z}_\ell\}$ is irreducible and positive recurrent.

We refer to $\{\boldsymbol{Z}_\ell\}$ as a 2-d reflecting random walk and to the distributions $\{p_{ij}^{(k)}; i,j = 0, \pm 1\}$ as its modeling primitives. We shall represent them by various families of generating functions:

$$p_{i*}^{(k)}(u_2) = \sum_{j=0,\pm 1} u_2^j p_{i,j}^{(k)}, \qquad p_{*j}^{(k)}(u_1) = \sum_{i=0,\pm 1} u_1^i p_{ij}^{(k)},$$

for $i, j = 0, \pm 1$, and

$$p_{**}^{(k)}(u_1, u_2) = \sum_{i,j=0,\pm 1} u_1^i u_2^j p_{ij}^{(k)},$$

for $u_1, u_2 > 0$ and $k = +, 0, 1, 2$. In the sequel, we shall denote $p_{ij}^{(+)}$ simply by p_{ij}.

2.2 QBD formulation

Alternatively, we may formulate the 2-d reflecting random walk as a QBD process. For this, we take the first coordinate n_1 of the state $(n_1, n_2) \in S$ as the level, and n_2 as the background state, and write that $\{\boldsymbol{Z}_\ell\}$ has the transition probability matrix

$$P = \begin{pmatrix} B_0 & B_1 & 0 & 0 & \cdots \\ A_{-1} & A_0 & A_1 & 0 & \cdots \\ 0 & A_{-1} & A_0 & A_1 & \ddots \\ 0 & 0 & A_{-1} & A_0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2.1}$$

where the submatrices A_i and B_i are given by

$$A_i = \begin{pmatrix} p_{i0}^{(1)} & p_{i1}^{(1)} & 0 & \cdots \\ p_{i(-1)} & p_{i0} & p_{i1} & \cdots \\ 0 & p_{i(-1)} & p_{i0} & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad B_i = \begin{pmatrix} p_{i0}^{(0)} & p_{i1}^{(0)} & 0 & \cdots \\ p_{i(-1)}^{(2)} & p_{i0}^{(2)} & p_{i1}^{(2)} & \cdots \\ 0 & p_{i(-1)}^{(2)} & p_{i0}^{(2)} & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Define the rate matrix R for the Markov additive kernel $\{A_i; i = 0, \pm 1\}$ as the minimal nonnegative solution of the equation:

$$R = A_1 + RA_0 + R^2 A_{-1}.$$

and define $A_*(z) = z^{-1}A_{-1} + A_0 + zA_1$, then the following fact is well known; it is immediate from the Wiener-Hopf factorization (see Miyazawa and Zhao [19] and Miyazawa and Zwart [20]), and its direct proof can be found in [15].

Lemma 2.1 *Take $\boldsymbol{x} > \boldsymbol{0}$ and $\eta \in (0,1)$. One has $\boldsymbol{x}R = \eta\boldsymbol{x}$ if and only if $\boldsymbol{x}A_*(\eta^{-1}) = \boldsymbol{x}$.* □

The significance of this lemma is due to the fact that R is a major component of the stationary distribution, as we recall in the next lemma.

Lemma 2.2 *Let $\boldsymbol{\pi} \equiv \{\pi(i,j); i,j \in S\}$ be a probability measure on S, partitioned as $\boldsymbol{\pi} = \begin{bmatrix}\boldsymbol{\pi}_0 & \boldsymbol{\pi}_1 & \boldsymbol{\pi}_2 & \ldots\end{bmatrix}$, with $\boldsymbol{\pi}_n = \begin{bmatrix}\pi(n,0) & \pi(n,1) & \ldots\end{bmatrix}$, for $n \geq 0$. It is the stationary measure of $\{\boldsymbol{Z}_\ell\}$ if and only if the following conditions hold.*

$$\boldsymbol{\pi}_0 = \boldsymbol{\pi}_0 B_0 + \boldsymbol{\pi}_1 A_{-1}, \tag{2.2}$$

$$\boldsymbol{\pi}_1 = \boldsymbol{\pi}_0 B_1 + \boldsymbol{\pi}_1 A_0 + \boldsymbol{\pi}_2 A_{-1}, \tag{2.3}$$

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_1 R^{n-1}, \qquad n \geq 2. \tag{2.4}$$

The proof of this lemma may be found in Neuts [21]. We note here that $H \equiv (I, R, R^2, \ldots)$ is an occupation measure of the Markov additive process with kernel $\{A_i; i = 0, \pm 1\}$. Hence, we must have (2.4), while (2.2) and (2.3) are direct consequences of the stationary equation.

For the most part, we consider the first coordinate of $\boldsymbol{Z}_\ell$ to be the level, but we might as well use the second coordinate and develop a thoroughly parallel line of arguments. We distinguish the corresponding results by using † as a superscript; thus, $A_i^\dagger$ is equivalent to A_i with the $p_{ij}^{(1)}$s replaced by $p_{ij}^{(2)}$s, and $B_i^\dagger$ is equivalent to B_i, with the superscripts "2" replaced by "1".

3 Conditions for product-form

We now determine when a probability distribution $\boldsymbol{\pi} \equiv \{\pi(i,j)\}$ has product-form, that is, when it is expressed as

$$\pi(i,j) = \mu_i \nu_j, \qquad i,j \in \mathbb{Z}_+, \tag{3.1}$$

for some probability vectors $\boldsymbol{\mu} \equiv \{\mu_i; i \in \mathbb{Z}_+\}$ and $\boldsymbol{\nu} \equiv \{\nu_i; i \in \mathbb{Z}_+\}$, or $\boldsymbol{\pi} = \boldsymbol{\mu} \otimes \boldsymbol{\nu}$, where $\otimes$ is the Kronecker product. This equation is equivalent to

$$\boldsymbol{\pi}_n = \mu_n \boldsymbol{\nu}, \qquad n = 0, 1, \ldots \tag{3.2}$$

The first result is expressed in terms of the QBD formulation.

Theorem 3.1 *The stationary distribution* $\boldsymbol{\pi}$ *of* $\{\boldsymbol{Z}_\ell\}$ *has the product-form* $\boldsymbol{\pi} = \boldsymbol{\mu} \otimes \boldsymbol{\nu}$ *if and only if either one of the following two conditions holds.*
C.1 *There exists* $\eta_1 \in (0, 1)$ *such that*

$$\boldsymbol{\nu} A_*(\eta_1^{-1}) = \boldsymbol{\nu}, \tag{3.3}$$
$$\mu_0 \boldsymbol{\nu} B_1 = \mu_1 \eta_1^{-1} \boldsymbol{\nu} A_1. \tag{3.4}$$
$$\mu_0 \boldsymbol{\nu}(I - B_0) = \mu_1 \boldsymbol{\nu} A_{-1}. \tag{3.5}$$

In that case,

$$\mu_n = \mu_1 \eta_1^{n-1}, \qquad n \geq 1. \tag{3.6}$$

C.2 *There exists* $\eta_2 \in (0, 1)$ *such that*

$$\boldsymbol{\mu} A_*^\dagger(\eta_2^{-1}) = \boldsymbol{\mu}, \tag{3.7}$$
$$\nu_0 \boldsymbol{\mu} B_1^\dagger = \nu_1 \eta_2^{-1} \boldsymbol{\mu} A_1^\dagger. \tag{3.8}$$
$$\nu_0 \boldsymbol{\mu}(I - B_0^\dagger) = \nu_1 \boldsymbol{\mu} A_{-1}^\dagger. \tag{3.9}$$

In that case,

$$\nu_n = \nu_1 \eta_2^{n-1}, \qquad n \geq 1. \tag{3.10}$$

PROOF. We prove that C.1 is necessary and sufficient. It follows from (2.4) and (3.2) that the ratio μ_n/μ_{n-1} must be constant for $n \geq 2$. Denoting it by η_1, we have (3.6). We also have $\boldsymbol{\nu} R = \eta_1 \boldsymbol{\nu}$, which together with Lemma 2.1 implies (3.3). Substituting (3.2) into (2.2) and (2.3) yields (3.5) and (3.4). Thus, (3.3–3.6) are necessary.

Conversely, (3.3–3.5) implies that $\boldsymbol{\nu} R = \eta_1 \boldsymbol{\nu}$ and $\boldsymbol{\pi}_n$ of (3.2) satisfies (2.2), (2.3) and (2.4), so that $\boldsymbol{\pi}$ is the stationary distribution if ν_0 and μ_0 are appropriately chosen.

The equivalence of C.1 and C.2 is obvious because of the symmetric form of $\boldsymbol{\pi}$. □

Remark 3.2 Equation (3.3) may have infinitely many positive solutions $\boldsymbol{\nu}$, which are fully obtained in Theorem 3.1 of Miyazawa [17]. It is somehow surprising that C.1 is equivalent to C.2, and therefore C.1 implies (3.10), which uniquely determines $\boldsymbol{\nu}$.

Henceforth, we know that the product-form stationary distribution, if it exists, is characterized by the six parameters η_1, μ_0, μ_1, η_2, ν_0 and ν_1. The following fact is immediate from Theorem 3.1.

Corollary 3.3 *Either one of the conditions C.1 and C.2 of Theorem 3.1 implies that*

$$\pi(n_1, n_2) = \mu_1 \nu_1 \eta_1^{n_1-1} \eta_2^{n_2-1}, \qquad n_1, n_2 \geq 1. \tag{3.11}$$

An immediate question is whether one might have $\pi(n_1, n_2) = \mu_0\nu_0\eta_1^{n_1}\eta_2^{n_2}$ for all $n_1, n_2 \geq 0$, in other words, one wonders whether or not $\mu_1 = \mu_0\eta_1$ and $\nu_1 = \nu_0\eta_2$. Such is the case for the discrete time version of the Jackson network with two nodes: then, the marginal distributions $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are geometric; this, however, is a special case only. The following corollary answers the question in general.

We write that the vector $\boldsymbol{x} = (x_0.x_1, \ldots)$ is geometric if there exists a constant $a > 0$ such that $x_n = x_0 a^n$ for all $n \geq 1$, we write that $\boldsymbol{x}$ is modified-geometric if $x_n = x_1 a^{n-1}$ for $n \geq 1$, but possibly with $x_1 \neq x_0 a$.

Corollary 3.4 *If the stationary distribution has the product-form (3.11), then $\mu_1 = \mu_0\eta_1$, and $\boldsymbol{\mu}$ is geometric, if and only if $\boldsymbol{\nu}$ satisfies*

$$\boldsymbol{\nu} B_1 = \boldsymbol{\nu} A_1. \tag{3.12}$$

Similarly, $\boldsymbol{\nu}$ is geometric if and only if $\boldsymbol{\mu} B_1^\dagger = \boldsymbol{\mu} A_1^\dagger$.

Proof. Since the stationary distribution has product-form, we have C.1 of Lemma 3.1. Suppose $\boldsymbol{\mu}$ is geometric, then $\mu_n = \mu_0\eta_1^n$, and therefore (3.4) implies (3.12). On the other hand, if (3.12) holds, then (3.4) implies that $\mu_1 = \mu_0\eta_1$. □

In view of this discussion, we may classify the product-form distribution (3.1) into four cases:

- Both $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are geometric.
- $\boldsymbol{\mu}$ is geometric, and $\boldsymbol{\nu}$ is modified-geometric with $\nu_1 \neq \nu_0\eta_2$.
- $\boldsymbol{\mu}$ is modified-geometric with $\mu_1 \neq \mu_0\eta_1$, and $\boldsymbol{\nu}$ is geometric.
- Both $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are modified-geometric.

We are particularly interested in the first class because of its simplicity, but we start with the general case.

Our next task is to replace the vector-matrix expressions in (3.3–3.5) and (3.7–3.9) by closed form expressions in terms of the modeling primitives. These will be easier to check, as we show later.

3.1 Characterization with the modeling primitives

First, we focus on the eigenvector equations (3.3, 3.10) on the one hand, (3.6, 3.7) on the other hand. We shall give in Section 3.3 a geometric interpretation for the conditions of the following lemma.

Lemma 3.5 *For η_1 and η_2 in $(0,1)$, and ν_0 and $\nu_1 > 0$, we have $\boldsymbol{\nu} A_*(\eta_1^{-1}) = \boldsymbol{\nu}$ with $\boldsymbol{\nu} = \begin{bmatrix}\nu_0 & \nu_1 & \nu_1\eta_2 & \nu_1\eta_2^2 & \ldots\end{bmatrix}$ if and only if the following conditions hold:*

$$p_{**}(\eta_1^{-1}, \eta_2^{-1}) = 1, \tag{3.13}$$

$$p_{**}^{(1)}(\eta_1^{-1}, \zeta_2^{-1}) = 1, \tag{3.14}$$

$$\nu_0(1 - p_{*0}^{(1)}(\eta_1^{-1})) = \nu_1 p_{*(-1)}(\eta_1^{-1}), \tag{3.15}$$

where

$$\zeta_2 = p_{*1}(\eta_1^{-1})/(\eta_2 p_{*(-1)}(\eta_1^{-1})). \tag{3.16}$$

Similarly, $\boldsymbol{\mu} A_*^{\dagger}(\eta_2^{-1}) = \boldsymbol{\mu}$ with $\boldsymbol{\mu} = \begin{bmatrix}\mu_0 & \mu_1 & \mu_1\eta_1 & \mu_1\eta_1^2 & \ldots\end{bmatrix}$ for some η_1 and η_2 in $(0,1)$, and μ_0 and $\mu_1 > 0$, if and only if (3.13) holds, in addition to

$$p_{**}^{(2)}(\zeta_1^{-1}, \eta_2^{-1}) = 1, \tag{3.17}$$

$$\mu_0(1 - p_{0*}^{(2)}(\eta_2^{-1})) = \mu_1 p_{(-1)*}(\eta_2^{-1}) \tag{3.18}$$

where

$$\zeta_1 = p_{1*}(\eta_2^{-1})/(\eta_1 p_{(-1)*}(\eta_2^{-1}). \tag{3.19}$$

Remark 3.6 It is not hard to see that the equations of this lemma imply that

$$p_{**}(\eta_1^{-1}, \zeta_2^{-1}) = 1, \tag{3.20}$$

$$p_{**}(\zeta_1^{-1}, \eta_2^{-1}) = 1. \tag{3.21}$$

As we will see in Section 3.3, these equations have nice geometric interpretations.

PROOF. We have $\boldsymbol{\nu} A_*(\eta_1^{-1}) = \boldsymbol{\nu}$ for the modified-geometric vector $\boldsymbol{\nu}$ if and only if

$$\begin{aligned}
\nu_1 p_{*1}(\eta_1^{-1}) + \nu_1\eta_2 p_{*0}(\eta_1^{-1}) + \nu_1\eta_2^2 p_{*(-1)}(\eta_1^{-1}) &= \nu_1\eta_2 \\
\nu_0 p_{*1}^{(1)}(\eta_1^{-1}) + \nu_1 p_{*0}(\eta_1^{-1}) + \nu_1\eta_2 p_{*(-1)}(\eta_1^{-1}) &= \nu_1 \\
\nu_0 p_{*0}^{(1)}(\eta_1^{-1}) + \nu_1 p_{*(-1)}(\eta_1^{-1}) &= \nu_0
\end{aligned}$$

Observe that the first of these three equations is identical to (3.13) and the third is identical to (3.15), so that the system of equations above is equivalent to the system formed by (3.13, 3.15) and

$$\nu_0\eta_2 p_{*1}^{(1)}(\eta_1^{-1}) = \nu_1 p_{*1}(\eta_1^{-1}). \tag{3.22}$$

Consider the quadratic equation

$$p_{*(-1)}(\eta_1^{-1})z^2 + (p_{*0}(\eta_1^{-1}) - 1)z + p_{*1}(\eta_1^{-1}) = 0 \tag{3.23}$$

or $z(p_{**}(\eta_1^{-1}, z^{-1}) - 1) = 0$. Clearly, (3.13) requires that η_2 be one of its solutions, and ζ_2 defined in (3.16) is the second one. This justifies (3.20).

Using the definition of ζ_2, we transform (3.22) to

$$\nu_1 p_{*(-1)}(\eta_1^{-1}) = \nu_0 \zeta_2^{-1} p_{*1}^{(1)}(\eta_1^{-1})$$

and, combining it with (3.15), we obtain (3.14). We observe that (3.22) is not equivalent to (3.14), but the system (3.15, 3.22) is globally equivalent to the system (3.14, 3.15).

This concludes the first part of the lemma. The proof of the second part is identical and we do not detail it here. $\square$

Remark 3.7 Observe that the two roots η_2 and ζ_2 of (3.23) may happen to be equal. That is the case if

$$(1 - p_{*0}(\eta_1^{-1}))^2 - 4p_{*(-1)}(\eta_1^{-1})p_{*1}(\eta_1^{-1}) = 0,$$

and then $\eta_2 = \zeta_2 = (1 - p_{*0}(\eta_1^{-1}))/(2p_{*(-1)}(\eta_1^{-1}))$. This is the boundary case between the 2nd and 3rd cases in (3.37).

Remark 3.8 The conditions of this lemma are not sufficient for the distribution to have product-form: we still need to deal with the constraints (3.4) and (3.5), or (3.8) and (3.9) and we do this in the next theorem. The situation here is to be contrasted with a similar characterization for the two-dimensional semimartingale reflecting Brownian motion, SRBM for short. Dai and Miyazawa [6] show that conditions corresponding to those in this lemma are both necessary and sufficient for the SRBM to have a product-form stationary distribution. This difference between the two processes should not be surprising because the two-dimensional SRBM has a much simpler reflection mechanism.

We have already observed that there is some redundancy among the collection of equations that we manipulate. We identify an interesting minimal set in the theorem below, an important point being that we bring together the two major boundary sets through the equations (3.14) and (3.17) which, in the formulation of Lemma 3.5, belong to separate sets of conditions. This will be useful in the discussion later of our geometric interpretation.

Theorem 3.9 *For the stationary distribution to have product-form, it is necessary and sufficient that there exist η_1 and η_2 both in $(0,1)$ such that (3.13, 3.14 and 3.17) hold, and in addition*

$$p_{00}^{(0)} + \delta_2 p_{0(-1)}^{(2)} + \delta_1 p_{(-1)0}^{(1)} + \delta_1\delta_2 p_{(-1)(-1)} = 1, \tag{3.24}$$

$$\delta_2 - (p_{01}^{(0)} + \delta_2(p_{00}^{(2)} + \eta_2 p_{0(-1)}^{(2)})) = \delta_1(p_{(-1)1}^{(1)} + \delta_2(p_{(-1)0} + \eta_2 p_{(-1)(-1)})), \tag{3.25}$$

$$\eta_1(p_{10}^{(0)} + \delta_2 p_{1(-1)}^{(2)}) = \delta_1(p_{10}^{(1)} + \delta_2 p_{1(-1)}), \tag{3.26}$$

$$\eta_1(p_{11}^{(0)} + \delta_2(p_{10}^{(2)} + \eta_2 p_{1(-1)}^{(2)})) = \delta_1(p_{11}^{(1)} + \delta_2(p_{10} + \eta_2 p_{1(-1)})), \tag{3.27}$$

$$\delta_1 = (1 - p_{0*}^{(2)}(\eta_2^{-1}))/p_{-1,*}(\eta_2^{-1}), \tag{3.28}$$

$$\delta_2 = (1 - p_{*0}^{(1)}(\eta_1^{-1}))/p_{*(-1)}(\eta_1^{-1}), \tag{3.29}$$

where δ_1 and δ_2 are defined as

$$\delta_1 = \mu_1/\mu_0, \qquad \delta_2 = \nu_1/\nu_0.$$

Furthermore, $\boldsymbol{\mu}$ is geometric if and only if $\delta_1 = \eta_1$, and $\boldsymbol{\nu}$ is geometric if and only if $\delta_2 = \eta_2$.

Proof. By Theorem 3.1, a necessary and sufficient condition is given by the set of equations (3.3, 3.4 and 3.5). By the first part of Lemma 3.5, (3.3) is equivalent to (3.13, 3.14, 3.15), the latter being written here as (3.29), using the definition of δ_2.

Secondly, (3.5) is equivalent to (3.24, 3.25) and

$$(\eta_2 - (p_{01}^{(2)} + \eta_2 p_{00}^{(2)} + \eta_2^2 p_{0(-1)}^{(2)})) = \delta_1(p_{(-1)1} + \eta_2 p_{(-1)0} + \eta_2^2 p_{(-1)(-1)}),$$

this last equation being equivalent to (3.28).

Finally, using the definition of δ_1, one readily verifies that (3.4), once written in component details, is equivalent to the three equations (3.26, 3.27) and

$$(p_{11}^{(2)} + \eta_2 p_{10}^{(2)} + \eta_2^2 p_{1(-1)}^{(2)}) = \delta_1\eta_1^{-1}(p_{11} + \eta_2 p_{10} + \eta_2^2 p_{1(-1)}).$$

This last equation is successively rewritten as

$$\eta_1 p_{1*}^{(2)}(\eta_2^{-1}) = \delta_1 p_{1*}(\eta_2^{-1})$$

by definition of the generating functions, or

$$p_{1*}^{(2)}(\eta_2^{-1}) = \delta_1\zeta_1 p_{(-1)*}(\eta_2^{-1}) \tag{3.30}$$

by the definition (3.19) of ζ_1. The equations (3.28, 3.30) in turn imply (3.17), and this completes our identification of a coherent set of conditions. The proof of the remaining statements is immediate. $\square$

Remark 3.10 By the symmetry of the modeling structure, (3.25–3.27) may be replaced by

$$\delta_1 - (p_{10}^{(0)} + \delta_1(p_{00}^{(1)} + \eta_1 p_{(-1)0}^{(1)}))$$

$$= \delta_2(p_{1(-1)}^{(2)} + \delta_1(p_{0(-1)} + \eta_1 p_{(-1)(-1)})) \tag{3.31}$$

$$\eta_2(p_{01}^{(0)} + \delta_1 p_{(-1)1}^{(1)}) = \delta_2(p_{01}^{(2)} + \delta_1 p_{(-1)1}) \tag{3.32}$$

$$\eta_2(p_{11}^{(0)} + \delta_1(p_{01}^{(1)} + \eta_1 p_{(-1)1}^{(1)})) = \delta_2(p_{11}^{(2)} + \delta_1(p_{01} + \eta_1 p_{(-1)1})) \tag{3.33}$$

in the statement of Theorem 3.9.

3.2 Sufficient conditions for geometric product-form

Theorem 3.9 gives nine independent conditions for the product-form in terms of the modeling primitives $\{p_{i,j}\}$ and $\{p_{i,j}^{(k)}\}$, $k = 0,\ 1,\ 2$, for a total of 21 parameters. Thus, we are left with twelve degrees of freedom in the choice of modeling parameters that satisfy these conditions, and their interpretation is not very clear, even though we can in principle check them numerically. In this subsection, we restrict the model to have a simpler structure and obtain intuitively appealing conditions.

Lemma 3.11 *Assume that $A_1 = B_1$ or, equivalently, that*

$$p_{1i}^{(0)} = p_{1i}^{(1)}, \quad i = 0, 1, \qquad p_{1j}^{(2)} = p_{1j}, \quad j = 0, \pm 1. \tag{3.34}$$

The stationary distribution has product-form if and only if there exist η_1 and η_2 in $(0, 1)$ such that (3.13, 3.14, 3.17, 3.24, 3.25 and 3.29) hold, with $\delta_1 = \eta_1$. In that case, $\boldsymbol{\mu}$ is geometric.

Proof. This is a straightforward consequence of Theorem 3.9. With (3.34), it is easy to see that (3.26) and (3.27) reduce to $\delta_1 = \eta_1$, so that $\mu_1 = \eta_1 \mu_0$ and $\boldsymbol{\mu}$ is geometric.

Furthermore, (3.17) may be written as

$$\begin{aligned} & \zeta_1^{-1} p_{1*}^{(2)}(\eta_2^{-1}) = 1 - p_{0*}^{(2)}(\eta_2^{-1}) & \\ \text{or} \quad & \zeta_1^{-1} p_{1*}(\eta_2^{-1}) = 1 - p_{0*}^{(2)}(\eta_2^{-1}) & \text{since } A_1 = B_1, \\ \text{or} \quad & \eta_1 p_{(-1)*}(\eta_2^{-1}) = 1 - p_{0*}^{(2)}(\eta_2^{-1}) & \text{by definition of } \zeta_1, \end{aligned}$$

which is equivalent to (3.28) since $\eta_1 = \delta_1$. Thus, (3.28) is redundant and may be removed from the set of constraints. □

Obviously, a similar property holds if $A_1^\dagger = B_1^\dagger$; in that case, $\boldsymbol{\nu}$ is geometric and $\delta_2 = \eta_2$. If both $A_1 = B_1$ and $A_1^\dagger = B_1^\dagger$, things simplify even further, and we have the following theorem.

Theorem 3.12 *Assume that* $A_1 = B_1$ *and* $A_1^\dagger = B_1^\dagger$ *or, equivalently, that*

$$\begin{aligned} &p_{11} = p_{11}^{(0)} = p_{11}^{(1)} = p_{11}^{(2)}, \quad p_{10}^{(0)} = p_{10}^{(1)}, \quad p_{01}^{(0)} = p_{01}^{(2)}, \\ &p_{1j}^{(2)} = p_{1j}, \quad p_{j1}^{(1)} = p_{j1}, \quad j = 0, -1. \end{aligned} \tag{3.35}$$

The stationary distribution has product-form if and only if there exist η_1 *and* η_2 *in* $(0,1)$ *such that (3.13, 3.14, 3.17) hold, together with*

$$p_{00}^{(0)} + \eta_1 p_{(-1)0}^{(1)} + \eta_2 p_{0(-1)}^{(2)} + \eta_1\eta_2 p_{(-1)(-1)} = 1. \tag{3.36}$$

In that case, both $\boldsymbol{\mu}$ *and* $\boldsymbol{\nu}$ *are geometric.*

Proof. We start from the statement of Lemma 3.11 and show that, under the added assumption that $A_1^\dagger = B_1^\dagger$, (3.25) and (3.29) are redundant and may be omitted, we also show that (3.36) is equivalent to (3.24). This last claim is obvious as $\delta_1 = \eta_1$ and $\delta_2 = \eta_2$ by Lemma 3.11 and the comment which follows.

In order to prove that (3.29) may be removed from the list of constraints, we write (3.14) as

$$\begin{aligned} & \zeta_2^{-1} p_{*1}^{(1)}(\eta_1^{-1}) = 1 - p_{*0}^{(1)}(\eta_1^{-1}) && \\ \text{or} \quad & \zeta_2^{-1} p_{*1}(\eta_1^{-1}) = 1 - p_{*0}^{(1)}(\eta_1^{-1}) && \text{since } A_1^\dagger = B_1^\dagger, \\ \text{or} \quad & \eta_2 p_{*(-1)}(\eta_1^{-1}) = 1 - p_{*0}^{(1)}(\eta_1^{-1}) && \text{by the definition (3.16) of } \zeta_2, \\ \text{or} \quad & \delta_2 p_{*(-1)}(\eta_1^{-1}) = 1 - p_{*0}^{(1)}(\eta_1^{-1}) && \text{since } \delta_2 = \eta_2, \end{aligned}$$

which is equivalent to (3.29).

Finally, we write (3.17) as

$$\begin{aligned} & \zeta_1^{-1} p_{1*}^{(2)}(\eta_2^{-1}) = 1 - p_{0*}^{(2)}(\eta_2^{-1}) && \\ \text{or} \quad & \zeta_1^{-1} p_{1*}(\eta_2^{-1}) = 1 - p_{0*}^{(2)}(\eta_2^{-1}) && \text{since } A_1 = B_1, \\ \text{or} \quad & \eta_1 p_{(-1)*}(\eta_2^{-1}) = 1 - p_{0*}^{(2)}(\eta_2^{-1}) && \text{by the definition (3.19 of } \zeta_1, \end{aligned}$$

which is identical to (3.25) since $p_{(-1)1} = p_{(-1)1}^{(1)}$ and $p_{01}^{(2)} = p_{01}^{(0)}$. □

3.3 Geometric interpretation

We give in this section a geometric interpretation of the product-form conditions of Theorem 3.12. We follow the notation system of Miyazawa [17], and we switch from generating functions to moment generating functions. This

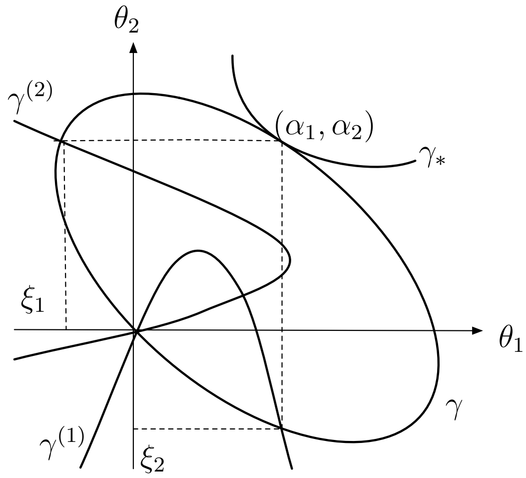

Figure 1: Geometric representation of the constraints for product-form.

is because moment generating functions are convex and therefore their level sets are convex, making them easier to handle. Thus, we define

$$\gamma(\boldsymbol{\theta}) = p_{**}(e^{\theta_1}, e^{\theta_2}), \qquad \gamma^{(k)}(\boldsymbol{\theta}) = p_{**}^{(k)}(e^{\theta_1}, e^{\theta_2}),$$

and

$$\gamma_*(\boldsymbol{\theta}) = p_{00}^{(0)} + e^{-\theta_1} p_{(-1)0}^{(1)} + e^{-\theta_2} p_{0(-1)}^{(2)} + e^{-\theta_1-\theta_2} p_{(-1)(-1)},$$

with $\boldsymbol{\theta} \equiv (\theta_1, \theta_2) \in \mathbb{R}^2$. Then, γ, $\gamma^{(k)}$ and γ_* are the moment generating functions corresponding to p_{**}, $p_{**}^{(k)}$ and the left-hand side of (3.36). It is notable that the curves:

$$\gamma(\boldsymbol{\theta}) = 1, \qquad \gamma^{(k)}(\boldsymbol{\theta}) = 1, \qquad \gamma_*(\boldsymbol{\theta}) = 1$$

are boundaries of convex sets. In particular, $\gamma(\boldsymbol{\theta}) = 1$ is a closed curve.

We give a hypothetical example on Figure 1 to illustrate Theorem 3.12. The curve marked with "γ" is the set of points such that $\gamma(\boldsymbol{\theta}) = 1$ and the other curves are similarly defined; also, $\alpha_k = -\log \eta_k$ and $\xi_k = -\log \zeta_k$, for $k = 1$ and 2.

In general, the geometric depiction is more involved, and we summarize below the presentation in [17] (see also Kobayashi and Miyazawa [13]). The object of these papers is to determine the exponential decay rates α_1 and α_2 of the marginal stationary distributions in the coordinate directions, which are defined as

$$\alpha_1 = -\lim_{n\to\infty} \frac{1}{n} \log \sum_{i=0}^{\infty} \pi(n, i), \qquad \alpha_2 = -\lim_{n\to\infty} \frac{1}{n} \log \sum_{i=0}^{\infty} \pi(i, n),$$

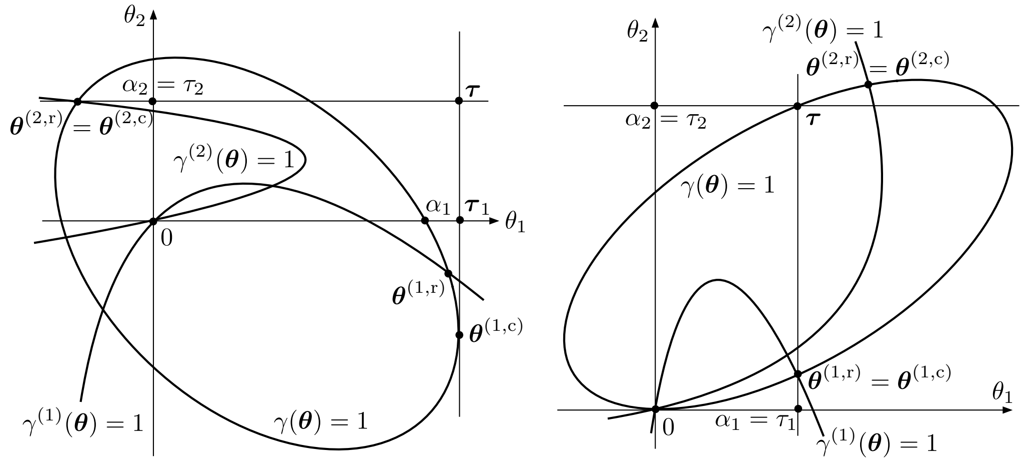

Figure 2: Visualisation of the auxiliary quantities used to determine the decay rates.

as long as they exists. There are two reasons for us to repeat the procedure of [17]. Firstly, it is helpful to see how the product-form conditions arise in the context of the 2-d reflecting random walk. Secondly, those decay rates will be used for approximations in Section 4.

We need to define a number of auxiliary points. The first are

$$\boldsymbol{\theta}^{(k,\mathrm{r})} = \arg\max_{(\theta_1,\theta_2)} \{\theta_k; \gamma(\boldsymbol{\theta}) = \gamma^{(k)}(\boldsymbol{\theta}) = 1\}, \qquad k = 1, 2,$$

see Figure 2. It is shown in [13] that, if $\theta_k^{(k,\mathrm{r})} \le 0$ for $k = 1$ or $k = 2$, then the stationary distribution either does not exist or does not have a light tail. Thus, as long as $\theta_k^{(k,\mathrm{r})} > 0$ for $k = 1, 2$, the stationary distribution exists and has light tails in all directions.

Next, we define

$$\boldsymbol{\theta}^{(k,\mathrm{c})} = \arg\max_{(\theta_1,\theta_2)} \{\theta_k; \gamma(\boldsymbol{\theta}) = 1, \gamma^{(k)}(\boldsymbol{\theta}) \le 1\}.$$

For the example on the left of Figure 2, $\theta^{(1,\mathrm{c})} \ne \theta^{(1,\mathrm{r})}$ and $\theta^{(2,\mathrm{c})} = \theta^{(2,\mathrm{r})}$.

To compute the decay rate, we define $\boldsymbol{\tau} \equiv (\tau_1, \tau_2)$ as

$$\boldsymbol{\tau} = \begin{cases} (\overline{f}_1(\theta_2^{(2,\mathrm{c})}), \theta_2^{(2,\mathrm{c})}) & \boldsymbol{\theta}^{(2,\mathrm{c})} < \boldsymbol{\theta}^{(1,\mathrm{c})}, \\ (\theta_1^{(1,\mathrm{c})}, \overline{f}_2(\theta_1^{(1,\mathrm{c})})) & \boldsymbol{\theta}^{(1,\mathrm{c})} < \boldsymbol{\theta}^{(2,\mathrm{c})}, \\ (\theta_1^{(1,\mathrm{c})}, \theta_2^{(2,\mathrm{c})}) & \text{otherwise}, \end{cases} \tag{3.37}$$

where $\overline{f}_1$ and $\overline{f}_2$ are obtained as

$$\overline{f}_1(\theta_2) = \max\{\theta; \gamma(\theta, \theta_2) = 1\}, \qquad \overline{f}_2(\theta_1) = \max\{\theta; \gamma(\theta_1, \theta) = 1\}.$$

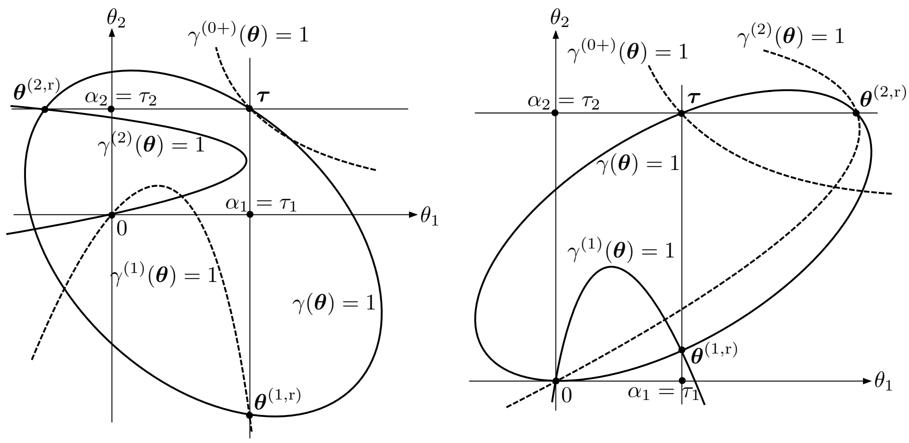

Figure 3: Modification of Figure 2 for the product-form of cases (3a) and (3b), where the dashed curves are modified ones.

The vector $\boldsymbol{\tau}$ must be positive because $\theta_k^{(k,\mathrm{c})} \geq \theta_k^{(k,\mathrm{r})} > 0$. It is shown in [17] that

$$\lim_{n\to\infty} \frac{1}{n} \log \pi(n,i) = -\tau_1, \qquad \lim_{n\to\infty} \frac{1}{n} \log \pi(i,n) = -\tau_2$$

for any $i \geq 0$. Hence, τ_1 and τ_2 are also decay rates but not necessarily for the marginal distributions. Then, by Theorem 3.3 of [13], the decay rates α_1 and α_2 for the marginal distributions are obtained as

$$\alpha_1 = \begin{cases} \tau_1, & \overline{f}_2(\tau_1) \geq 0, \\ \beta_1, & \overline{f}_2(\tau_1) < 0. \end{cases} \quad \alpha_2 = \begin{cases} \tau_2, & \overline{f}_1(\tau_2) \geq 0, \\ \beta_2, & \overline{f}_1(\tau_2) < 0, \end{cases} \tag{3.38}$$

where β_1 (or β_2) is the positive solution x of $\gamma(x,0) = 1$ (or $\gamma(0,x) = 1$). Thus, $\boldsymbol{\alpha}$ may be different from $\boldsymbol{\tau}$. Obviously, (α_1, α_2) must be identical with $\boldsymbol{\tau}$ if the stationary distribution has product-form.

The above procedure to get the decay rates may look complicated. However, it is actually not so hard at least numerically.

There are three typical cases for the product-form according to whether

(3a) $\boldsymbol{\tau} = (\theta_1^{(1,\mathrm{r})}, \theta_2^{(2,\mathrm{r})})$,

(3b) $\boldsymbol{\tau} = (\theta_1^{(1,\mathrm{r})}, \overline{f}(\theta_2^{(2,\mathrm{r})}))$,

(3c) $\boldsymbol{\tau} = (\overline{f}_1(\theta_2^{(2,\mathrm{r})}), \theta_2^{(2,\mathrm{r})})$.

Since (3b) and (3c) are symmetric with respect to the coordinate labels, we draw pictures for (3a) and (3b) only in Figure 3.

Figure 3 is obtained from Figure 2 after modifying γ_* and γ_1 in the left panel, γ_* and γ_2 in the right panel. It is instructive to examine the dashed lines, corresponding to the modified generating functions, and to observe how the pictures are specialized in the case of product-form.

As we show in the next section, the geometric view is helpful to find a good product-form approximation. It also suggests a way to find stochastic upper and lower bounds for the stationary distribution: if we shrink the curves for γ, $\gamma^{(1)}$ and $\gamma^{(2)}$, and suitably change the one for $\gamma^{(0+)}$ so that the sufficient conditions of Theorem 3.12 are satisfied, then we can expect to have a product-form upper bound in stochastic order. The verification, however, requires further study and is left as an open problem.

4 Product-form approximations

Exponential decay rates for the stationary distribution of multidimensional processes have been well studied in the literature (see, Miyazawa [18]), but no further information is available except for simple cases. Thus, one often looks for approximations using a simple function. The advantage of product-form approximations is to reduce a two-dimensional distribution to one dimensional.

Traditionally, product-form approximations have been suggested in ad hoc manners, and their theoretical supports have been scarcely considered (see, e.g., [10, 11]). We take a different point of view here.

Our idea is to change the model parameters in such a way that the new system has product-form and remains close to the original one, in some sense to be specified. We have three theorems and one lemma at our disposal: Theorems 3.1, 3.9, Lemma 3.11 (all three have two versions, depending on which boundary phase is emphasized) and Theorem 3.12.

In [15], it is assumed that $A_1 = B_1$ and the approach is based on Theorem 3.1: a matrix B_0 is constructed such that (3.5) holds, with $\boldsymbol{\nu}$ and η_1 defined by (3.3). The major drawback is that the transition structure is thoroughly changed at the boundary faces S_0 and S_2.

Here, we want to preserve the random walk "nearest neighbors" transition structure, and we rely on the other characterizations of product-form. We use a tilde for the original process to distinguish it from the approximating random walk. For example, a product-form approximation of the stationary distribution is denoted by $\boldsymbol{\pi}$, it must have the form:

$$\pi(i,j) = \mu_1 \nu_1 \eta_1^{i-1} \eta_2^{j-1}, \qquad i,j \geq 1,$$

by Corollary 3.3. We define the objective function G as

$$G(P) = \sum_{i,j=0,\pm 1} (p_{ij} - \tilde{p}_{ij})^2 + \sum_{k=0}^{2} \sum_{i,j=0,\pm 1} (p_{ij}^{(k)} - \tilde{p}_{ij}^{(k)})^2. \tag{4.1}$$

We minimize G subject to the smallest number of necessary conditions for the Markov chain with transition matrix P to have a product-form stationary distribution. Each of these constraints may be considered to be at most a quadratic polynomial in each variable (η_k or δ_k for instance) if the other variables are fixed. Thus, we may easily use a package software like Mathematica to numerically solve the optimization problem.

We have considered two approximations, with different sets of constraints.

APw — weaker constraints. We minimize $G(P)$ under the product-form constraints of Theorems 3.9 or 3.12, depending on the circumstances. In this way we are certain to obtain a product-form random walk where the transition probabilities are closest to the original ones.

APs — stronger constraints. We compute the decay rates α_1, α_2 of the original model and require, in addition to the constraints from Theorems 3.9 or 3.12, that

$$\eta_1 = e^{-\alpha_1}, \qquad \eta_2 = e^{-\alpha_1}.$$

This approximation keeps the exact decay rates in the coordinate directions.

It may happen that *APs* is not feasible because the constraints are too strong, and define an empty set of values for the parameters. If that is the case, we would go back to solving *APw*. This is still useful, for it would tell us how the model has to be changed in order to obtain product-form.

Example 4.1 — Non-product-form network

This is a queueing network with two nodes, labeled 1 and 2. In what follows, we do not use tildes for the original variables because they can be easily distinguished in the context. There are three mutually independent Poisson arrival streams, with rates λ_1, λ_2 and λ_s, respectively. The process with rate λ_i, for $i = 1$ or 2, is dedicated to node i and the process with rate λ_s corresponds to simultaneous arrivals to both nodes. Service times are exponentially distributed, with rate μ_i at node i. The routing probability from node i to j is r_{ij}, where $j = 0$ stands for the outside.

The traffic intensity ρ_i at node i is given by

$$\rho_i = \frac{\lambda_i + \lambda_s + r_{i(3-i)}(\lambda_{3-i} + \lambda_s)}{(1 - r_{12}r_{21})\mu_i}, \qquad i = 1, 2.$$

The parameters are chosen as follows:

$$\begin{array}{llll} \lambda_1 = 0.2, & \lambda_2 = 0.2, & \lambda_s = 0.2, & \\ r_{10} = 0.8, & r_{12} = 0.2, & r_{20} = 0.8, & r_{21} = 0.2, \end{array}$$

and we take $\mu_1 = \mu_2 = \mu$ so that ρ_1 and ρ_2 are both equal to 0.5. This queueing network is formulated as a discrete-time reflecting random walk by uniformization, its transition probabilities p_{ij} are given in the row labeled "Model" in Table 1.

Prob.	p_{11}	p_{10}	$p_{1(-1)}$	p_{01}	p_{00}	$p_{0(-1)}$	$p_{(-1)1}$,	$p_{(-1)0}$
Model	0.077	0.077	0.077	0.077	0.	0.308	0.077	0.308
APs	0.037	0.092	0.118	0.092	0.	0.271	0.118	0.271
APw	0.037	0.067	0.090	0.067	0.002	0.319	0.090	0.319
Prob.	$p_{(-1)(-1)}$	$p^{(1)}_{10}$	$p^{(1)}_{00}$	$p^{(1)}_{(-1)0}$	$p^{(2)}_{01}$	$p^{(2)}_{00}$	$p^{(2)}_{0(-1)}$	$p^{(0)}_{00}$
Model	0.	0.077	0.385	0.308	0.077	0.385	0.308	0.769
APs	0.	0.118	0.379	0.255	0.118	0.379	0.255	0.727
APw	0.010	0.105	0.396	0.305	0.105	0.396	0.305	0.752

Table 1: Parameters for the reflecting random walk (Model) and the approximations *APs* and *APw*; missing probabilities are determined from (3.35).

It is an example of a system of parallel queues with simultaneous arrivals, and its stationary distribution is known to be hard to analyze (see, Flatto and Hahn [9]). Obviously, $A_1 = B_1$ and $A_1^\dagger = B_1^\dagger$ and we may restrict ourselves to the constraints expressed in Theorem 3.12. From the drawing on the left of Figure 4, we see that it does not have product-form. We have applied the approximation procedures *APs* and *APw* and obtained the new transition probabilities given in the rows *APs* and *APw* of Table 1.

The effect of the stronger constraints may be seen on the right of Figure 4. The decay rates are preserved, the point (α_1, α_2) has not changed, but it is now on the curves $\gamma(\boldsymbol{\theta}) = 1$ and $\gamma_*(\boldsymbol{\theta}) = 1$. The effect of *APw* is shown on Figure 5.

As *APs* has the stronger constraints, its mean square error, equal to 0.0190, is larger than that of *APw*, equal to 0.0046. In that sense, *APw* is better than *APs*. However, Figures 4 and 5 show that the curves of *APs* fit much better those of the original model. Precisely, *APs* largely changes the

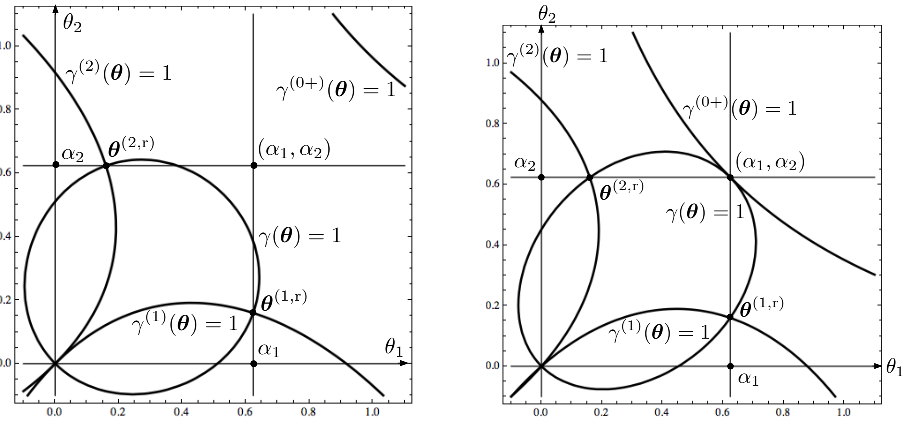

Figure 4: The left picture shows the original model, and the right picture shows the result of *APs*.

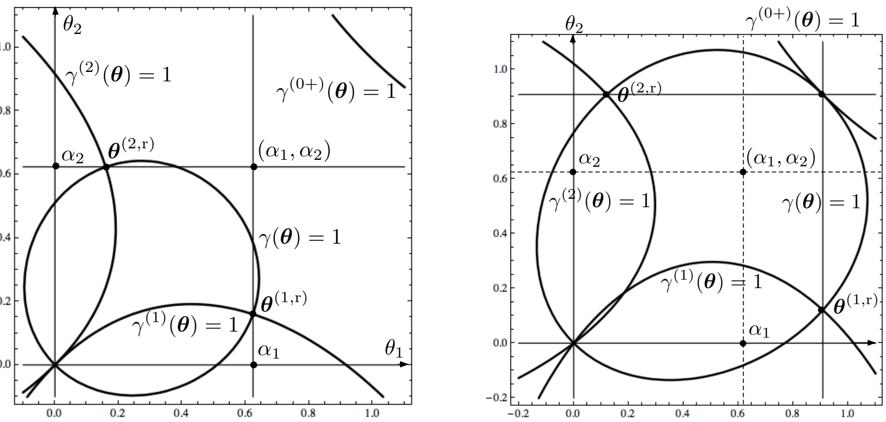

Figure 5: The left picture shows the original model, and the right picture shows the result of *APw*.

boundary condition $\gamma_*(\boldsymbol{\theta}) = 1$ and not much the other, while *APw* changes all the parameters in a balanced way.

We next compare the mean and variance of the marginal distributions in the first coordinate direction, for the original model and its two product-form approximations. We give them on Figure 6 as functions of ρ_i , for $\rho_1 = 0.25$ to 0.9 with step 0.05. The QBD approximations are obtained by formulating the reflecting random walk as the quasi-birth-and-death process with transition matrix (2.1) the level and background states are number of customers in the first and second nodes, and the second coordinate is truncated at a sufficiently large value, so that the QBD approximations can be considered very close to exact values. As we can see, *APs* is much better

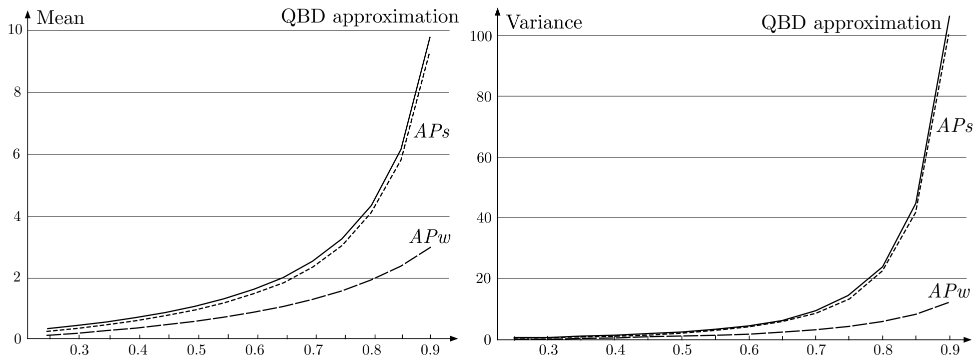

Figure 6: The left panel is for the mean queue length of node 1 and the right panel is for its variance, where the horizontal axis is the traffic intensity.

than APw in those characteristics.

Thus, minimizing with fewer constraints the square error of the modeling primitives may not give a better approximation, and we may conclude that APs is safer than APw. In other words, the exact decay rates would be useful information for approximation. Of course, this is a suggestion based on our special example only, and there should be a large number of numerical tests and further study for other possible approximations. For example, we may also use the decay rates other than those in the coordinate directions. We hope the present work stimulates those investigations in the future.

Acknowledgement

This paper was initiated while Masakiyo Miyazawa visited Guy Latouche at the Université Libre de Bruxelles. We thank the Ministère de la Communauté française de Belgique for its support through the ARC grant AUWB-08/13–ULB 5. We also thank Mr. Yusuke Fukui for numerical computations for the product-form and QBD approximations in Section 4. This research is supported in part by Japan Society for the Promotion of Science under grant No. 24310115.

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