Asymptotic Behavior of Loss Rate for Feedback Finite Fluid Queue with Downward Jumps

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Abstract

We consider a feedback finite fluid queue with downward jumps, where the net flow rate and the jump size for the fluid are controlled by a background Markov chain with a finite state space. The *finite* means that the fluid queue has a finite buffer. The *feedback* means that the transition structure of the background process may change when the buffer content becomes empty or full. In this paper, we show that the loss rate for the fluid queue decays exponentially as the buffer size becomes large under a negative drift condition.

Keywords: Feedback fluid queue, finite buffer, downward jumps, loss rate, Markov modulated, asymptotic behavior.

1 Introduction

We are concerned with a fluid queue, which consists of a server and a buffer. When the input rate of the fluid flow exceeds the processing capacity of the server, the unprocessed fluid is stored in the buffer. The input and output rates of the fluid flow depend on the state of a background process. Assume that the background process is a continuous time Markov chain on a finite state space. Fluid queues has been applied in many situations, e.g., real-world systems such that petroleum and chemical industries, performance analysis of high-speed data networks, designing computer systems, and so on. Anick et al. [1] study the data-handling switch in a computer network by using a fluid queue with an infinite buffer. Bonald [4] and van Foreest et al. [8] study the performance evaluation of TCP/IP (see e.g., [5]) by using fluid queues. Specifically, [8] studies the feature of *Additive Increase / Multiplicative Decrease* in TCP/IP by using a feedback fluid queue with a finite buffer. The *feedback* means that the transition structure of the background Markov chain may change when the buffer content becomes empty or full. da Silva Soares et al. [6] show that the stationary density of the buffer content for the feedback fluid queue with a finite buffer is expressed as a linear combination of two exponential matrices, by using the matrix analytic method. We call this fluid queue a *feedback finite fluid queue (FFFQ, for short).*

In this study, we extend the FFFQ in such a way that an accumulated net fluid flow may have downward jumps, i.e., instantaneous draining, when the background state changes. The *downward jump* is motivated by the following observations. Consider a bottleneck router connected to TCP sources in the Internet. IP packets arriving from the TCP sources enter buffer of the router, and wait for being served. The router sends IP packets to output links according to a routing table, which may be timely updated. Assume that IP packets and the output links have various sizes and capacities, respectively. When IP packets of small sizes are transferred to the output link with low capacity, the buffer content of the router slowly decreases. On the other hand, when IP packets of large sizes are transferred to the output link with high capacity, the buffer content rapidly decreases, which may be regarded as *downward jumps*.

We aim to consider an asymptotic behavior of the loss rate $\ell_{\text{Loss}}^{(b)}$ for the FFFQ with downward jumps as the buffer size b goes to infinity. Under a negative drift condition, we show that there exist positive constants c and α such that

$$\lim_{b\to\infty} e^{\alpha b}\ell_{\text{Loss}}^{(b)} = c,$$

and we obtain α as the solution of a certain equation (see Theorem 5.1 in Section 5). Note that Asmussen et al. [2] study an asymptotic behavior of a Levy process with two reflecting boundaries, and generalize it to a Markov modulated Levy process. They show that the loss rate decays exponentially as the one of the boundaries goes to infinity. However, their model does not have the feedback mechanism.

In this study, we heavily use the results in [11] and [9]. In [11], a Markov modulated fluid queue with an infinite buffer and downward jumps is studied. It is shown there that the stationary distribution of the buffer content has a matrix exponential form, which is one of the key observations for our study (see Theorem 3.1 of [11]). In [9], a Markov modulated fluid queue with upward jumps is studied. For this model, the hitting probability for an upper level does not have the matrix exponential form, since this process is not skip free in the upward direction. The hitting probability is also the key observation for our study because of the two-sided reflections of our model. Ramaswami [12] studies a fluid flow model by using a quasi-birth-and-death (QBD, for short) process, and as mentioned in [6] and [7], the FFFQ has a close connection with a finite level QBD process. In this sense, if there is no jump, our results are related to those in [10], where a many server queue with a finite buffer is modeled by a finite level QBD process. The loss probability for the queueing model is shown to decay geometrically as the buffer size goes to infinity under a negative drift condition.

This paper is composed of six sections. In section 2, we introduce a Markov additive process with downward jumps. In section 3, we put a reflecting boundary at level 0 for this additive process. Then we obtain a feedback fluid queue with an infinite buffer and downward jumps. In section 4, we further put a reflecting boundary at level b, and obtain the FFFQ with downward jumps. In section 5, we give the asymptotic behavior of the loss rate for the FFFQ with downward jumps. Finally, we provide some numerical result in section 6.

2 MAP (Markov additive process) with downward jumps

When the two boundaries of the FFFQ with downward jumps are removed, we obtain a Markov additive process (MAP, for short) with downward jumps. So we first consider the MAP with downward jumps and its hitting probability for an upper level in this section. The additive process and its hitting probabilities play key roles in the subsequent sections. Before proceeding, we first introduce some notations for matrices and vectors, which will be used throughout the paper. Denote an identity matrix, a unit vector and a zero vector by I, $\mathbf{1}$ and $\mathbf{0}$, respectively, where their sizes can be identified in the contexts where they appear. For vector $\mathbf{a}$, let $\Delta_{\mathbf{a}}$ be the diagonal matrix whose (i,i)-th element is the i-th element of the vector $\mathbf{a}$. Denote the (i,j)-th element of matrix A by $[A]_{ij}$, and the i-th element of vector $\mathbf{a}$ by $[\mathbf{a}]_i$ unless stated otherwise. Let A^{T} be the transposition of matrix A.

Let $M(t)$ be a continuous time Markov chain (CTMC, for short) with a finite state space $\mathcal{S}$. The transition rate matrix of $M(t)$ is decomposed into two $\mathcal{S}\times\mathcal{S}$ matrices C and D, where C is ML-matrix and D is nonnegative matrix such that $(C+D)\mathbf{1}=\mathbf{0}$. Throughout the paper, assume that $C+D$ is irreducible. Then we have a stationary distribution π for $C+D$, i.e., $\pi(C+D)=\mathbf{0}$ and $\pi\mathbf{1}=1$. Let $\mathbf{r}=(r(i);i\in\mathcal{S})$, where $r(\cdot)$ is a real-valued function defined on $\mathcal{S}$. Define an additive process $X(t)$ driven by $M(t)$ as follows.

(i) When $M(t)=i(\in\mathcal{S})$, $X(t)$ changes at rate $r(i)$, i.e., $\frac{d}{dt}X(t)=r(M(t))$.

(ii) When $M(t)$ changes from i to j by $[C]_{ij}$, the changing rate of $X(t)$ changes from $r(i)$ to $r(j)$.

(iii) When $M(t)$ changes from i to j by $[D]_{ij}$, the changing rate of $X(t)$ changes from $r(i)$ to $r(j)$, and $X(t)$ jumps down with a jump size subject to a distribution F_{ij}.

The two-dimensional CTMC $(X(t),M(t))$ with a state space $(-\infty,\infty)\times\mathcal{S}$ is called a *MAP with downward jumps*, or simply called *MAP* (see [11]). We call the first component *level process* or sometimes *additive component*, and call the second component *background process*.

For simplicity, assume that r takes nonzero values. Divide the state space $\mathcal{S}$ into two disjoint subsets $\mathcal{S}^-$ and $\mathcal{S}^+$, where $\mathcal{S}^-=\{i\in\mathcal{S}|r(i)<0\}$ and $\mathcal{S}^+=\{i\in\mathcal{S}|r(i)>0\}$. To avoid the trivial case, assume that neither $\mathcal{S}^-$ nor $\mathcal{S}^+$ is null set. Then we partition π, $\mathbf{r}$, C and D according to $\mathcal{S}^-$ and $\mathcal{S}^+$ as:

$$\pi=(\pi^-,\pi^+),\quad \mathbf{r}=(\mathbf{r}^-,\mathbf{r}^+),\quad \begin{pmatrix} C^{--} & C^{-+} \\ C^{+-} & C^{++} \end{pmatrix},\quad \begin{pmatrix} D^{--} & D^{-+} \\ D^{+-} & D^{++} \end{pmatrix},$$

(see Figure 1).

For $x\geq 0$, let τ_x^+ be a first hitting time when the level process hits x, i.e., $\tau_x^+=\inf\{t>0;X(t)\geq x\}$. For $x\geq 0$, define the $\mathcal{S}\times\mathcal{S}^+$ matrix $R^{\bullet+}(x)$ whose (i,j)-th element is given by

$$[R^{\bullet+}(x)]_{ij}=\mathrm{P}(M(\tau_x^+)=j|X(0)=0,M(0)=i),$$

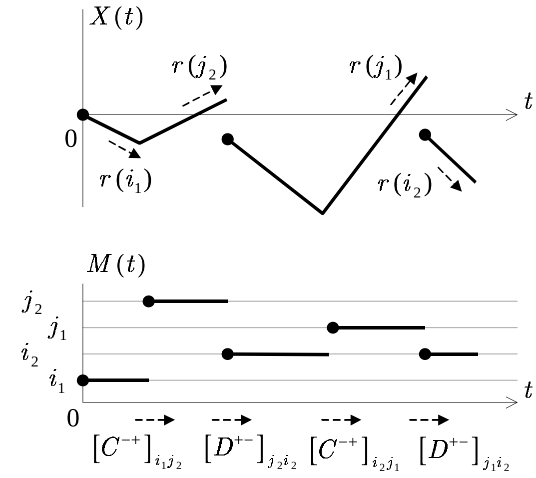

Figure 1: MAP with downward jumps, where $\mathcal{S}^- = \{i_1, i_2\}$, $\mathcal{S}^+ = \{j_1, j_2\}$.

which is a first hitting probability for an upper level x with a background state j, starting from level 0 with a background state i. Divide $R^{\bullet +}(x)$ into blocks $R^{-+}(x)$ and $R^{++}(x)$ according to $\mathcal{S}^-$ and $\mathcal{S}^+$. Throughout the paper, assume the following condition :

$$E[X(1) - X(0)] < 0, \tag{1}$$

which is referred to as a negative drift condition. For $x \geq 0$, define the $\mathcal{S} \times \mathcal{S}$ matrix $D(x) = ([D]_{ij} F_{ij}(x); i, j \in \mathcal{S})$. Then $R^{\bullet +}(x)$ has the matrix exponential form.

Proposition 2.1 (Theorem 3.1 of [11]) Under the negative drift condition (1), there exist the $\mathcal{S}^+ \times \mathcal{S}^+$ defective transition rate matrix U and the $\mathcal{S}^- \times \mathcal{S}^+$ substochastic matrix R_0 satisfying

$$\begin{pmatrix} R_0 \\ I \end{pmatrix} U = \Delta_{\mathbf{r}}^{-1} \left\{ C \begin{pmatrix} R_0 \\ I \end{pmatrix} + \int_0^\infty D(du) \begin{pmatrix} R_0 \\ I \end{pmatrix} \exp(uU) \right\}. \tag{2}$$

And we have

$$\begin{pmatrix} R^{-+}(x) \\ R^{++}(x) \end{pmatrix} = \begin{pmatrix} R_0 \\ I \end{pmatrix} \exp(xU), \quad x \geq 0. \tag{3}$$

Remark 2.1 R_0 and U are computed by the following recursion formula :

$$U_{[0]} = C^{++}, R_{[0]} = 0,$$

$$U_{[n+1]} = \Delta_{\mathbf{r}^+}^{-1} \left(C^{++} + C^{+-} R_{[n]} + \int_0^\infty (D^{+-}(du) R_{[n]} + D^{++}(du)) \exp(uU_{[n]}) \right),$$

$$R_{[n+1]} = -\Delta_{\mathbf{r}^-}^{-1} \Bigg(C^{-+} + (-\eta \Delta_{\mathbf{r}^-} + C^{--}) R_{[n]} + \int_0^\infty (D^{--}(du) R_{[n]} + D^{-+}(du)) \exp(uU_{[n]}) \Bigg) (\eta I - U_{[n]})^{-1}, \quad x \geq 0,$$

where $\eta (> 0)$ is chosen so that $-\eta \Delta_{\mathbf{r}^-} + C^{--}$ becomes a nonnegative matrix (see [11] and [13]).

We next introduce the two hitting times,

$$\tau_0^- = \inf\{t > 0 | X(t) \le 0\}, \quad \tau_b^+ = \inf\{t > 0 | X(t) \ge b\},$$

where $b > 0$. Let ${}_0A_{0b}^{++}$ be the $\mathcal{S}^+ \times \mathcal{S}^+$ matrix whose (i, j)-th element is given by

$$[{}_0A_{0b}^{++}]_{ij} = \mathrm{P}(M(\tau_b^+) = j, \tau_b^+ < \tau_0^- | X(0) = 0, M(0) = i).$$

This is the hitting probability that the level process hits level b with a background state $j \in \mathcal{S}^+$ before it goes below level 0, starting from level 0 with a background state $i \in \mathcal{S}^+$. Let P_{00}^{++} be the $\mathcal{S}^+ \times \mathcal{S}^+$ matrix whose (i, j)-th element is the probability that $X(t)$ returns to level 0 with a background state $j \in \mathcal{S}^+$, starting from level 0 with a background state $i \in \mathcal{S}^+$. That is, the (i, j)-th element of P_{00}^{++} is given by

$$[P_{00}^{++}]_{ij} = \mathrm{P}(M(\zeta_0^+) = j | X(0) = 0, M(0) = i),$$

where $\zeta_0^+ = \inf\{t > 0; X(t-) < 0 < X(t+)\}$. The following result plays key role for our main result. We defer its proof to Appendix.

Lemma 2.1 Let $-\alpha(< 0)$ be the Perron Frobenius (P-F, for short) eigenvalue of the defective transition rate matrix U, and $\mathbf{q}^+$ be the corresponding positive right eigenvector, i.e., $U\mathbf{q}^+ = -\alpha\mathbf{q}^+$. Under the negative drift condition (1), we have

$$\lim_{b \to \infty} e^{\alpha b} {}_0A_{0b}^{++} = (I - P_{00}^{++})\mathbf{q}^+\mathbf{u}^+\Delta_{\mathbf{q}^+}^{-1},$$

where $\mathbf{u}^+$ is the stationary distribution of the non-defective transition rate matrix $\Delta_{\mathbf{q}^+}^{-1}(\alpha I + U)\Delta_{\mathbf{q}^+}$. Furthermore, $-\alpha$ is obtained as the solution of

$$\chi(z) = 0, \tag{4}$$

where $\chi(z)$ is the P-F eigenvalue of the following ML-matrix:

$$C + \int_0^\infty D(du)\exp(zu) - z\Delta_{\mathbf{r}}.$$

By Proposition 2.1, the hitting probability for an upper level has the matrix exponential form. In general, the hitting probability for a lower level does not have the similar form because of the downward jumps. However, from [9], we know that the hitting probability for a lower level is obtained by integrating the matrix exponential form. In what follows, we present this result. For $x > 0$, let $H^{+\bullet}(x) = (H^{+-}(x), H^{++}(x))$ be the $\mathcal{S}^+ \times \mathcal{S}$ matrix whose (i, j)-th element is given by

$$[H^{+\bullet}(x)]_{ij} = \mathrm{P}(M(\tau_0^-) = j, X(\tau_0^-) \in (-x, 0) | X(0) = 0, M(0) = i),$$

which is the first hitting probability for a lower level with a jump. Let H_0^{+-} be the $\mathcal{S}^+ \times \mathcal{S}^-$ matrix whose (i, j)-th element is given by

$$[H_0^{+-}]_{ij} = \mathrm{P}(M(\tau_0^-) = j, X(\tau_0^-) = 0 | X(0) = 0, M(0) = i),$$

which is the first hitting probability for a lower level without jump. These hitting probabilities for a lower level are given as follows.

Proposition 2.2 (Lemma 3.1 of [9]) The hitting probability for a lower level with a jump is given by

$$(H^{+-}(x), H^{++}(x)) = \Delta_{\mathbf{r}^+}^{-1}\Delta_{\pi^+}^{-1}\left\{\int_0^x ds \int_s^\infty \Delta_\pi \tilde{D}(dy) \begin{pmatrix} \tilde{R}_0 \exp((y-s)\tilde{U}) \\ \exp((y-s)\tilde{U}) \end{pmatrix}\right\}^{\mathrm{T}},$$

where $\tilde{D}(y) = \Delta_\pi^{-1} D(y)^{\mathrm{T}} \Delta_\pi$. $\tilde{R}_0$ and $\tilde{U}$ are the $\mathcal{S}^- \times \mathcal{S}^+$ and $\mathcal{S}^+ \times \mathcal{S}^+$ matrices, respectively, satisfying

$$\begin{pmatrix} \tilde{R}_0 \\ I \end{pmatrix} \tilde{U} = \Delta_{\mathbf{r}}^{-1}\left\{\tilde{C}\begin{pmatrix} \tilde{R}_0 \\ I \end{pmatrix} + \int_0^\infty \tilde{D}(dy)\begin{pmatrix} \tilde{R}_0 \\ I \end{pmatrix}\exp(y\tilde{U})\right\},$$

where $\tilde{C} = \Delta_\pi^{-1} C^{\mathrm{T}} \Delta_\pi$. On the other hand, the hitting probability without jump is given by

$$H_0^{+-} = \Delta_{\mathbf{r}^+}^{-1}\Delta_{\pi^+}^{-1}\int_0^\infty ds \int_0^\infty \left\{\Delta_{\pi^-} K^{--}(s)\tilde{W}^{-\bullet}(dy)\begin{pmatrix} \tilde{R}_0 \\ I \end{pmatrix}\exp((s+y)\tilde{U})\right\}^{\mathrm{T}},$$

where $\tilde{W}^{-\bullet}(y)$ is the $\mathcal{S}^- \times \mathcal{S}$ matrix defined by $\Delta_{\pi^-}^{-1} W^{\bullet -}(y)^{\mathrm{T}} \Delta_\pi$. $W^{\bullet -}(y)$ is the $\mathcal{S} \times \mathcal{S}^-$ matrix whose (i,j)-th element is given by

$$1_{\{i \neq j\}}[C]_{ij}\delta(y) + [D(y)]_{ij},$$

where 1_A is the indicator function for event A and $\delta(y)$ is the Dirac distribution which has a unit mass at the origin. $K^{--}(s)$ is the $\mathcal{S}^- \times \mathcal{S}^-$ diagonal matrix whose (k,k)-th element is given by $\exp(c(k)s/r(k))$, where

$$c(k) = -[C]_{kk}.$$

Remark 2.2 $\tilde{R}_0$ and $\tilde{U}$ are obtained by the similar formula as noted in Remark 2.1.

By the negative drift condition (1), the following $\mathcal{S}^+ \times \mathcal{S}$ matrix

$$(H_0^{+-} + H^{+-}, H^{++}) \tag{5}$$

is stochastic, where $H^{+u} = \lim_{x\to\infty} H^{+u}(x)$ for $u = \pm 1$.

3 FIFQ (Feedback infinite fluid queue) with downward jumps

In this section, we set a boundary to the MAP $(X(t), M(t))$ so that the additive component is reflected at level 0. Consider a two-dimensional CTMC $(Y(t), J(t))$ with a state space $[0,\infty) \times \mathcal{S}$, where its transition structure is given as follows (see Figure 2).

(i) While $Y(t) > 0$, $(Y(t), J(t))$ has the same transition structure as the MAP $(X(t), M(t))$.

(ii) When $Y(t)$ hits level 0, the transition rate matrix of $J(t)$ immediately changes to another $\mathcal{S}^- \times \mathcal{S}$ matrix :

$$\underline{C} = \left(\underline{C}^{--} \quad \underline{C}^{-+} \right),$$

where $\underline{C}^{--}$ is the $\mathcal{S}^- \times \mathcal{S}^-$ ML-matrix, $\underline{C}^{-+}$ is the $\mathcal{S}^- \times \mathcal{S}^+$ non-negative matrix and $\underline{C}\mathbf{1} = \mathbf{0}$. This modification for the transition structure of the background process $J(t)$ is referred to as *feedback*. There are two types of the hitting level 0.

(iia) If it occurs due to C^{--} or D^{--}, $Y(t)$ stays in level 0 until $J(t)$ changes due to $\underline{C}^{-+}$. After $J(t)$ changes due to $\underline{C}^{-+}$, $Y(t)$ goes up from level 0. Then $(Y(t), J(t))$ again has the same transition structure as (i).

(iib) If it occurs due to D^{-+}, $Y(t)$ immediately goes up from level 0. Then $(Y(t), J(t))$ again has the same transition structure as (i).

We introduce a nonnegative (resp. positive) valued function r_{in} (resp. r_{out}) defined on $\mathcal{S}$. Assume that there is a fluid input (resp. output) at rate $r_{\text{in}}(i)$ (resp. $r_{\text{out}}(i)$) when $J(t) = i (\in \mathcal{S})$. In this paper, the net flow rate r is given by the difference of the input and output rates, i.e.,

$$r = r_{\text{in}} - r_{\text{out}}.$$

Then $(Y(t), J(t))$ is referred to as a *feedback infinite fluid queue (FIFQ, for short) with downward jumps*, or simply referred to as *FIFQ*.

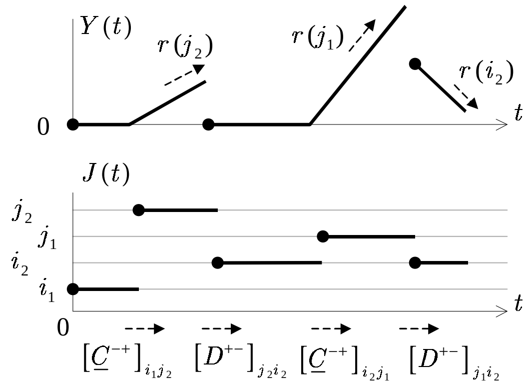

Figure 2: FIFQ with downward jumps, where $\mathcal{S}^- = \{i_1, i_2\}$, $\mathcal{S}^+ = \{j_1, j_2\}$.

By the negative drift condition (1), there exists a stationary distribution for $(Y(t), J(t))$.

Proposition 3.1 (Theorem 4.1 of [11]) For $x > 0$, we have

$$\mathrm{P}(Y > x, J = i) = \begin{cases} [\Delta_{\pi^-} R_0 \exp(xU)\mathbf{1}]_i\,, & i \in \mathcal{S}^-, \\ [\Delta_{\pi^+} \exp(xU)\mathbf{1}]_i\,, & i \in \mathcal{S}^+, \end{cases}$$

where (Y, J) means $(Y(t), J(t))$ in steady state.

Let $\mathbf{p}^-$ be the $\mathcal{S}^-$-dimensional row vector whose i-th element is given by

$$[\mathbf{p}^-]_i = \mathrm{P}(Y = 0, J = i).$$

By censoring $(Y(t), J(t))$ at subspace $\{0\} \times \mathcal{S}^-$, $\mathbf{p}^-$ is obtained as a stationary measure for $Q_{00} = \underline{C}^{--} + \underline{C}^{-+}(I - H^{++})^{-1}(H_0^{+-} + H^{+-})$, that is,

$$\mathbf{p}^- Q_{00} = \mathbf{0}.$$

Note that Q_{00} is a non-defective transition rate matrix by the negative drift condition (1). For $x > 0$, let $\mathbf{p}(x)$ be the $\mathcal{S}$- dimensional row vector whose i-th element is given by

$$[\mathbf{p}(x)]_i = \mathrm{P}(Y > x, J = i).$$

Then $\mathbf{p}^-$ is normalized so that $\mathbf{p}^-\mathbf{1} + \mathbf{p}(0)\mathbf{1} = 1$, that is,

$$\mathbf{p}^-\mathbf{1} + \pi^- R_0 \mathbf{1} + \pi^+ \mathbf{1} = 1, \tag{6}$$

by Proposition 3.1.

4 FFFQ (Feedback finite fluid queue) with downward jumps

In this section, we put another boundary to the FIFQ with downward jumps $(Y(t), J(t))$ in such a way that the additive component is also reflected at level b, where $b > 0$. We further assume that the transition structure of the background process may change at level b. Denote this reflected additive process by $(Y^{(b)}(t), J^{(b)}(t))$, which is a two-dimensional CTMC with a state space $[0, b] \times \mathcal{S}$. The background process $J^{(b)}(t)$ has the following three types of transition structures depending on the level $Y^{(b)}(t)$ (see Figure 3).

(i) While $Y^{(b)}(t)$ stays in $(0, b)$, $(Y^{(b)}(t), J^{(b)}(t))$ has the same transition structure as FIFQ $(Y(t), J(t))$, i.e., $J^{(b)}(t)$ is a CTMC with transition rate matrix $C + D$.

(ii) When $Y^{(b)}(t)$ hits level b, the transition rate matrix of $J^{(b)}(t)$ immediately changes to another $\mathcal{S}^+ \times \mathcal{S}$ matrix :

$$\overline{C} + \overline{D} = \begin{pmatrix} \overline{C}^{+-} & \overline{C}^{++} \end{pmatrix} + \begin{pmatrix} \overline{D}^{+-} & \overline{D}^{++} \end{pmatrix},$$

where $\overline{C}^{++}$ is the $\mathcal{S}^+ \times \mathcal{S}^+$ ML-matrix, $\overline{C}^{+-}$, $\overline{D}^{+-}$ and $\overline{D}^{++}$ are the $\mathcal{S}^+ \times \mathcal{S}^-$, $\mathcal{S}^+ \times \mathcal{S}^-$ and $\mathcal{S}^+ \times \mathcal{S}^+$ nonnegative matrices, respectively, and $(\overline{C} + \overline{D})\mathbf{1} = \mathbf{0}$. After $Y^{(b)}(t)$ hits level b, there can be following three cases.

(iia) If $J^{(b)}(t)$ changes due to $\overline{C}^{++}$, $Y^{(b)}(t)$ stays in level b.

(iib) If $J^{(b)}(t)$ changes due to $\overline{C}^{+-}$, $Y^{(b)}(t)$ goes below level b and $(Y^{(b)}(t), J^{(b)}(t))$ again has the same transition structure as (i).

(iic) If $J^{(b)}(t)$ changes due to $\overline{D}$, $Y^{(b)}(t)$ jumps down below level b. The jump size is distributed subject to $\overline{D}(x)$, where $\overline{D}(x)$ is the $\mathcal{S}^+ \times \mathcal{S}$ matrix whose (i,j)-th element is $[\overline{D}]_{ij}G_{ij}(x)$, where $G_{ij}(x)$ is a distribution function. When the jump size is less than b, $(Y^{(b)}(t), J^{(b)}(t))$ has the same transition structure as (i). Otherwise, $Y^{(b)}(t)$ hits level 0 and $(Y^{(b)}(t), J^{(b)}(t))$ has the same transition structure as (iii).

(iii) When $Y^{(b)}(t)$ hits level 0, the transition rate matrix of $J^{(b)}(t)$ changes to $\underline{C} = (\underline{C}^{--}, \underline{C}^{-+})$. There are two types of the hitting level 0.

(iiia) If it occurs due to C^{--}, D^{--} or $\overline{D}^{+-}$, $Y^{(b)}(t)$ stays in level 0 until $J^{(b)}(t)$ changes due to $\underline{C}^{-+}$. After $J^{(b)}(t)$ changes due to $\underline{C}^{-+}$, $Y^{(b)}(t)$ goes up from level 0. Then $(Y^{(b)}(t), J^{(b)}(t))$ again has the same transition structure as (i).

(iiib) If it occurs due to D^{-+} or $\overline{D}^{++}$, $Y^{(b)}(t)$ immediately goes up from 0. Then $(Y^{(b)}(t), J^{(b)}(t))$ again has the same transition structure as (i).

This reflected additive process $(Y^{(b)}(t), J^{(b)}(t))$ is referred to as a *feedback finite fluid queue (FFFQ, for short) with downward jumps*, or simply referred to as *FFFQ*.

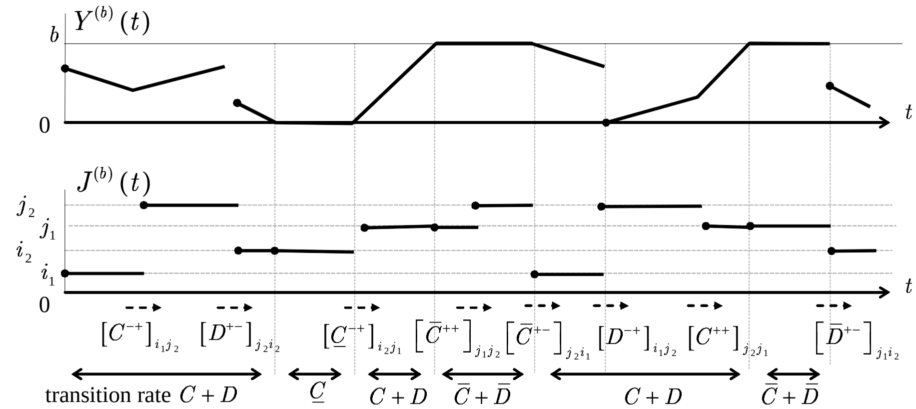

Figure 3: FFFQ with downward jumps, where $\mathcal{S}^- = \{i_1, i_2\}$, $\mathcal{S}^+ = \{j_1, j_2\}$.

5 Asymptotic behavior of loss rate for FFFQ with downward jumps

In this section, we study the asymptotic behavior of the loss rate $\ell^{(b)}_{\text{Loss}}$ for the FFFQ with downward jumps $(Y^{(b)}(t), J^{(b)}(t))$ as the buffer size b becomes large. Let $\mathbf{p}^{(b)}$ be the $\mathcal{S}$-dimensional probability vector whose i-th element is given by

$$[\mathbf{p}^{(b)}]_i = \begin{cases} \mathrm{P}(Y^{(b)} = 0, J^{(b)} = i), & i \in \mathcal{S}^-, \\ \mathrm{P}(Y^{(b)} = b, J^{(b)} = i), & i \in \mathcal{S}^+, \end{cases}$$

where $(Y^{(b)}, J^{(b)})$ means $(Y^{(b)}(t), J^{(b)}(t))$ in steady state. We partition $\mathbf{p}^{(b)}$ according to $\mathcal{S}^-$ and $\mathcal{S}^+$ such that $\mathbf{p}^{(b)} = (\mathbf{p}^{(b)-}, \mathbf{p}^{(b)+})$. Then the loss rate $\ell^{(b)}_{\text{Loss}}$ is given by

$$\ell^{(b)}_{\text{Loss}} = \mathbf{p}^{(b)+}\mathbf{r}^+_{\text{in}}, \tag{7}$$

where $\mathbf{r}_{\rm in}^{+} = (r_{\rm in}(i); i \in \mathcal{S}^{+})$. So it is sufficient to consider the asymptotic behavior of $\mathbf{p}^{(b)+}$ as b becomes large.

Note that $\mathbf{p}^{(b)}$ is a stationary measure for $(Y^{(b)}(t), J^{(b)}(t))$ censoring at subspace

$$\mathcal{S}_{\{0,b\}} = (\{0\} \times \mathcal{S}^{-}) \cup (\{b\} \times \mathcal{S}^{+}).$$

We introduce the following two hitting probabilities (see Figure 4). Let ${}_b\Psi_{x0}^{\bullet -}$ be the $\mathcal{S} \times \mathcal{S}^{-}$ matrix for $x \in [0, b]$ whose (i, j)-th element is given by

$$[{}_b\Psi_{x0}^{\bullet -}]_{ij} = \mathrm{P}(J^{(b)}(\tau_0^{(b)-}) = j, \tau_0^{(b)-} < \tau_b^{(b)+} | Y^{(b)}(0) = x, J^{(b)}(0) = i),$$

and ${}_0\Psi_{xb}^{\bullet +}$ be the $\mathcal{S} \times \mathcal{S}^{+}$ matrix for $x \in [0, b]$ whose (i, j)-th element is given by

$$[{}_0\Psi_{xb}^{\bullet +}]_{ij} = \mathrm{P}(J^{(b)}(\tau_b^{(b)+}) = j, \tau_b^{(b)+} < \tau_0^{(b)-} | Y^{(b)}(0) = x, J^{(b)}(0) = i),$$

where $\tau_0^{(b)-} = \inf\{t > 0 | Y^{(b)}(t) = 0, J^{(b)}(t) \in \mathcal{S}^{-}\}$ and $\tau_b^{(b)+} = \inf\{t > 0 | Y^{(b)}(t) = b, J^{(b)}(t) \in \mathcal{S}^{+}\}$. Partition ${}_b\Psi_{x0}^{\bullet -}$ and ${}_0\Psi_{xb}^{\bullet +}$ into blocks according to $\mathcal{S}^{-}$ and $\mathcal{S}^{+}$ such that

$${}_b\Psi_{x0}^{\bullet -} = \begin{pmatrix} {}_b\Psi_{x0}^{--} \\ {}_b\Psi_{x0}^{+-} \end{pmatrix}, \quad {}_0\Psi_{xb}^{\bullet +} = \begin{pmatrix} {}_0\Psi_{xb}^{-+} \\ {}_0\Psi_{xb}^{++} \end{pmatrix}.$$

Then the transition rate matrix for the censored process at subspace $\mathcal{S}_{\{0,b\}}$ is given by

$$Q = \begin{pmatrix} Q_{00}^{(b)} & Q_{0b}^{(b)} \\ Q_{b0}^{(b)} & Q_{bb}^{(b)} \end{pmatrix},$$

where the each submatrix is given by

$$\begin{aligned} Q_{00}^{(b)} &= \underline{C}^{--} + \underline{C}^{-+}{}_b\Psi_{00}^{+-}, \qquad Q_{0b}^{(b)} = \underline{C}^{-+}{}_0\Psi_{0b}^{++}, \\ Q_{b0}^{(b)} &= \overline{C}^{+-}{}_b\Psi_{b0}^{--} + \int_0^b \overline{D}^{+-}(dx){}_b\Psi_{(b-x)0}^{--} + \int_b^\infty \overline{D}^{+-}(dx) \\ &\quad + \int_0^b \overline{D}^{++}(dx){}_b\Psi_{(b-x)0}^{+-} + \int_b^\infty \overline{D}^{++}(dx){}_b\Psi_{00}^{+-}, \\ Q_{bb}^{(b)} &= \overline{C}^{++} + \overline{C}^{+-}{}_0\Psi_{bb}^{-+} + \int_0^b \overline{D}^{++}(dx){}_0\Psi_{(b-x)b}^{++} + \int_b^\infty \overline{D}^{++}(dx){}_0\Psi_{0b}^{++} \\ &\quad + \int_0^b \overline{D}^{+-}(dx){}_0\Psi_{(b-x)b}^{-+}. \end{aligned}$$

Then $\mathbf{p}^{(b)-}$ and $\mathbf{p}^{(b)+}$ satisfy

$$\mathbf{p}^{(b)-}Q_{00}^{(b)} + \mathbf{p}^{(b)+}Q_{b0}^{(b)} = \mathbf{0}, \quad \mathbf{p}^{(b)-}Q_{0b}^{(b)} + \mathbf{p}^{(b)+}Q_{bb}^{(b)} = \mathbf{0}. \tag{8}$$

By the negative drift condition (1), we have

$$\lim_{b\to\infty} {}_0\Psi_{0b}^{++} = 0, \quad \lim_{b\to\infty} Q_{bb}^{(b)} = \hat{Q}_{00},$$

where $\hat{Q}_{00}$ is a defective transition rate matrix. Since $\lim_{b\to\infty} Q_{0b}^{(b)} = 0$, we have

$$\lim_{b\to\infty} \mathbf{p}^{(b)+} = \mathbf{0}.$$

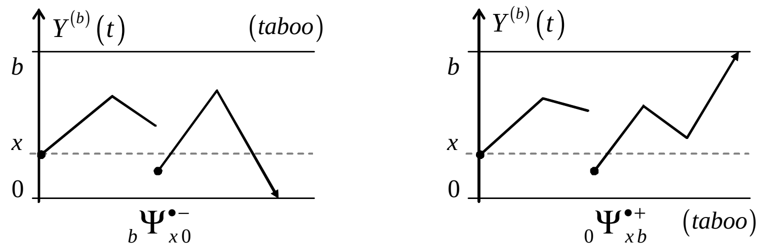

Figure 4: Hitting probabilities for FFFQ with downward jumps

Furthermore, we have

$$\lim_{b\to\infty} {}_b\Psi^{+-}_{00} = (I - H^{++})^{-1}(H^{+-}_0 + H^{+-}),$$

since the effect of level b disappears as b becomes large. Hence, (8) and the above observations imply that

$$\lim_{b\to\infty} \mathbf{p}^{(b)-} = \mathbf{p}^{-},$$

which is a stationary measure for Q_{00} with the normalizing condition (6).

By the second equation of (8), we have

$$\mathbf{p}^{(b)+}(-Q^{(b)}_{bb}) = \mathbf{p}^{(b)-}\underline{C}^{-+}\,{}_0\Psi^{++}_{0b}, \tag{9}$$

which implies that the asymptotic behavior of $\mathbf{p}^{(b)+}$ is determined by that of ${}_0\Psi^{++}_{0b}$. From Proposition 2.1, Proposition 2.2 and Lemma 2.1, we arrive at the main result. Its proof is deferred to Appendix.

Theorem 5.1 The asymptotic behavior of the loss rate is given by

$$\lim_{b\to\infty} e^{\alpha b}\ell^{(b)}_{\text{Loss}} \;=\; \mathbf{p}^{-}\underline{C}^{-+}(I - H^{++})^{-1}(I - P^{++}_{00})(\mathbf{q}^{+}\mathbf{u}^{+}\Delta^{-1}_{\mathbf{q}^{+}})(-\hat{Q}_{00})^{-1}\mathbf{r}^{+}_{\text{in}},$$

where $-\alpha$ is the solution of (4), and $\hat{Q}_{00}$, $P^{++}_{00}\mathbf{q}^{+}$ are obtained as follows.

(I) $\hat{Q}_{00} = \overline{C}^{++} + \overline{C}^{+-}R_0 + \int_0^\infty \overline{D}^{++}(dx)\exp(xU) + \int_0^\infty \overline{D}^{+-}(dx)R_0\exp(xU)$.

(II) $P^{++}_{00}\mathbf{q}^{+}$ is given by

$$\left\{H^{+-}_0 R_0 + \Delta^{-1}_{\mathbf{r}^{+}}\Delta^{-1}_{\pi^{+}}\int_0^\infty \left(\begin{pmatrix}\tilde{R}_0\\ I\end{pmatrix}\exp(y\tilde{U})V(y)\right)^{\mathrm{T}}\Delta_\pi D(dy)\begin{pmatrix}R_0\\ I\end{pmatrix}\right\}\mathbf{q}^{+}$$

where

$$V(y) = \Delta_{\tilde{\mathbf{q}}^{+}}(\mathbf{1}\kappa - \hat{U})^{-1}\left(\exp(-y\hat{U}) + y\mathbf{1}\kappa - I\right)\Delta^{-1}_{\tilde{\mathbf{q}}^{+}},$$

and $\tilde{\mathbf{q}}^{+}$ is the P-F right eigenvector for $\tilde{U}$ with P-F eigenvalue $-\alpha$. κ is the stationary distribution for $\hat{U} = \Delta^{-1}_{\tilde{\mathbf{q}}^{+}}(\alpha I + \tilde{U})\Delta_{\tilde{\mathbf{q}}^{+}}$.

6 Numerical examples

We provide some numerical examples for the FFFQ with downward jumps by computing the positive constants α and c such that

$$\lim_{b\to\infty} e^{\alpha b}\ell^{(b)}_{\mathrm{Loss}} = c.$$

This indicates that we may approximate the loss rate by $ce^{-\alpha b}$. Suppose that the jump sizes are deterministic for each possible transition. That is, let B be the $\mathcal{S}\times\mathcal{S}$ matrix, whose (i,j)-th element denotes the jump size of the buffer when the background state changes from i to j. Then $D(x)$ is given by

$$[D(x)]_{ij} = [D]_{ij}1_{\{x=[B]_{ij}\}}.$$

Similarly, let $\overline{B}^{+-}$ (resp. $\overline{B}^{++}$) be the $\mathcal{S}^+\times\mathcal{S}^-$ (resp. $\mathcal{S}^+\times\mathcal{S}^+$) matrix, whose (i,j)-th element denotes the jump size when the background state changes from i to j at level b. Then $\overline{D}^{+-}(x)$ and $\overline{D}^{++}(x)$ are given by

$$[\overline{D}^{+-}(x)]_{ij} = [\overline{D}^{+-}]_{ij}1_{\{x=[\overline{B}^{+-}]_{ij}\}}, \quad [\overline{D}^{++}(x)]_{ij} = [\overline{D}^{++}]_{ij}1_{\{x=[\overline{B}^{++}]_{ij}\}}.$$

Assume the following parameter settings:

$$\begin{aligned}
&\mathcal{S}^- = \{0,1\}, \quad \mathcal{S}^+ = \{2\},\\
&\begin{pmatrix} r_{\mathrm{in}}(0)\\ r_{\mathrm{in}}(1)\\ r_{\mathrm{in}}(2)\end{pmatrix} = \begin{pmatrix} 5.5\\ 8.0\\ 10.0\end{pmatrix}, \quad \begin{pmatrix} r_{\mathrm{out}}(0)\\ r_{\mathrm{out}}(1)\\ r_{\mathrm{out}}(2)\end{pmatrix} = \begin{pmatrix} 6.0\\ 8.7\\ 6.0\end{pmatrix},\\
&C = \begin{pmatrix} -4.6 & 1.5 & 2.3\\ 0.6 & -2.7 & 1.2\\ 0.5 & 0.8 & -2.1\end{pmatrix}, \quad D = \begin{pmatrix} 0.2 & 0.1 & 0.5\\ 0.3 & 0.4 & 0.2\\ 0.5 & 0.1 & 0.2\end{pmatrix}, \quad B = \begin{pmatrix} 2.3 & 1.3 & 1.5\\ 0.5 & 1.8 & 2.4\\ 2.4 & 5.0 & 2.1\end{pmatrix},\\
&\underline{C}^{--} = \begin{pmatrix} -3.0 & 1.0\\ 1.0 & -2.0\end{pmatrix}, \quad \underline{C}^{-+} = \begin{pmatrix} 2.0\\ 1.0\end{pmatrix},\\
&\overline{C}^{+-} = \begin{pmatrix} 1.5 & 1.3\end{pmatrix}, \quad \overline{C}^{++} = (-4.1), \quad \overline{D}^{+-} = \begin{pmatrix} 0.4 & 0.5\end{pmatrix}, \quad \overline{D}^{++} = (0.3),\\
&\overline{B}^{+-} = \begin{pmatrix} 1.1 & 5.0\end{pmatrix}, \quad \overline{B}^{++} = (2.7).
\end{aligned}$$

In this case, we have

$$\text{mean drift} = -0.046225 < 0, \quad \alpha = 0.012454, \quad c = 2.213758.$$

We further consider the following two cases.

(case1) Change the jump size due to D-transition : $[B]_{21} = 5.0 \to [B]_{21} = 15.0$. Then we have

$$\text{mean drift} = -0.546513 < 0, \quad \alpha = 0.077897, \quad c = 3.469491.$$

Since the jump size is increased when the additive component is below level b, the decay rate α considerably becomes larger.

(case2) Change the jump size due to $\overline{D}$-transition : $[\overline{B}]_1 = 5.0 \rightarrow [\overline{B}]_1 = 15.0$. Then we have

$$\text{mean drift} = -0.046225, \quad \alpha = 0.012454, \quad c = 1.394617.$$

Since the jump size is increased when the additive component stays in level b, only the prefactor c decreases.

Appendix

(Proof of Lemma 2.1) Consider the $\mathcal{S}^+ \times \mathcal{S}^+$ matrix ${}_bP_{00}^{++}$ whose (i,j)-th element is given by

$$[{}_bP_{00}^{++}]_{ij} = \mathrm{P}(M(\zeta_0^{(b)+}) = j | X(0) = 0, M(0) = i),$$

where $\zeta_0^{(b)+} = \inf\{t > 0; X(t-) < 0 < X(t+), X(u) < b, u \in (0,t)\}$ is the first time when the MAP $(X(t), M(t))$ crosses level 0 from below, avoiding level b (see Figure 5). Note

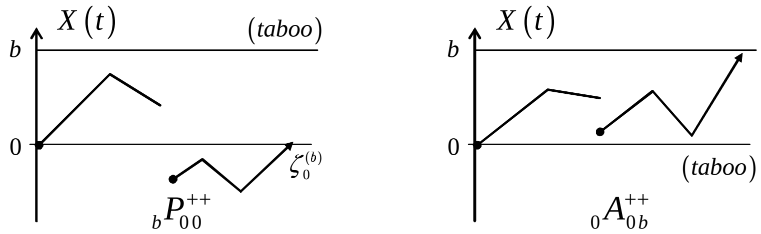

Figure 5: Hitting probabilities for MAP with downward jumps

that $\lim_{b\to\infty} {}_bP_{00}^{++} = P_{00}^{++}$, since the effect of level b disappears as b becomes large. By conditioning on the event that the MAP $(X(t), M(t))$ crosses the initial level from below for the first time, we have $R^{++}(b) = {}_0A_{0b}^{++} + {}_bP_{00}^{++}R^{++}(b)$. By Proposition 2.1, we have

$$ {}_0A_{0b}^{++} = (I - {}_bP_{00}^{++}) \exp(bU). \tag{10}$$

Since U is defective, there exists the P-F eigenvalue $-\alpha(< 0)$ and the corresponding positive right eigenvector $\mathbf{q}^+$. Note that $\Delta_{\mathbf{q}^+}^{-1}(\alpha I + U)\Delta_{\mathbf{q}^+}$ is a non-defective transition rate matrix. So it has the stationary distribution $\mathbf{u}^+$, i.e., $\mathbf{u}^+\Delta_{\mathbf{q}^+}^{-1}(\alpha I + U)\Delta_{\mathbf{q}^+} = \mathbf{0}$ and $\mathbf{u}^+\mathbf{1} = 1$. By the standard Markov chain theory, we have

$$\lim_{b\to\infty} \exp(b\Delta_{\mathbf{q}^+}^{-1}(\alpha I + U)\Delta_{\mathbf{q}^+}) = \mathbf{1}\mathbf{u}^+,$$

which is equivalent to

$$\lim_{b\to\infty} e^{\alpha b} \exp(bU) = \mathbf{q}^+\mathbf{u}^+\Delta_{\mathbf{q}^+}^{-1}. \tag{11}$$

Combining (10) with (11) yields

$$\lim_{b\to\infty} e^{\alpha b} {}_0A_{0b}^{++} = (I - P_{00}^{++})\mathbf{q}^+\mathbf{u}^+\Delta_{\mathbf{q}^+}^{-1}.$$

By postmultiplying $\mathbf{q}^+$ to (2), we have

$$\left(-\alpha I - \Delta_{\mathbf{r}}^{-1}\left(C + \int_0^\infty \exp(-\alpha u)D(du)\right)\right)\begin{pmatrix} I \\ R_0 \end{pmatrix}\mathbf{q}^+ = \mathbf{0},$$

which implies that $-\alpha$ is obtained as a solution of $\chi(z) = 0$.

(Proof of Theorem 5.1) Consider the hitting probability that the MAP $(X(t), M(t))$ jumps blow level 0 with a background state in $\mathcal{S}^+$, starting from level 0 with a background state in $\mathcal{S}^+$, avoiding level b. This is equivalent to the probability that the FFFQ $(Y^{(b)}(t), J^{(b)}(t))$ returns to level 0 while increasing, starting from level 0 with a background state in $\mathcal{S}^+$, avoiding level b. Let ${}_bH^{++}$ be the $\mathcal{S}^+ \times \mathcal{S}^+$ matrix whose (i,j)-th element is given by

$$[{}_bH^{++}]_{ij} = \mathrm{P}(M(\tau_0^-) = j, X(\tau_0^-) < 0, \tau_0^- < \tau_b^+ | X(0) = 0, M(0) = i).$$

Note that $\lim_{b\to\infty} {}_bH^{++} = H^{++}$. By conditioning on the event that $(Y^{(b)}(t), J^{(b)}(t))$ returns to level 0, we have ${}_0\Psi_{0b}^{++} = {}_0A_{0b}^{++} + {}_bH^{++}{}_0\Psi_{0b}^{++}$. Since ${}_bH^{++}$ is sub-stochastic, we have

$$ {}_0\Psi_{0b}^{++} = (I - {}_bH^{++})^{-1}{}_0A_{0b}^{++}. \tag{12}$$

From (9) and (12), we have $\mathbf{p}^{(b)+} = \mathbf{p}^{(b)-}\underline{C}^{-+}(I - {}_bH^{++})^{-1}{}_0A_{0b}^{++}(-Q_{bb}^{(b)})^{-1}$, which implies that

$$\lim_{b\to\infty} e^{\alpha b}\mathbf{p}^{(b)+} = \mathbf{p}^-\underline{C}^{-+}(I - H^{++})^{-1}(I - P_{00}^{++})(\mathbf{q}^+\mathbf{u}^+\Delta_{\mathbf{q}^+}^{-1})(-\hat{Q}_{00})^{-1} \tag{13}$$

by Lemma 2.1. In the following, we compute $\hat{Q}_{00}$, $\mathbf{p}^-$ and $P_{00}^{++}\mathbf{q}^+$ in the right side of (13). This completes the proof of Theorem 5.1,

(Proof of (I)) By the definition of $Q_{bb}^{(b)}$ and the dominated convergence theorem, we have

$$\hat{Q}_{00} = \overline{C}^{++} + \overline{C}^{+-}\hat{\Psi}_{00}^{-+} + \int_0^\infty \overline{D}^{++}(dx)\hat{\Psi}_{x0}^{++} + \int_0^\infty \overline{D}^{+-}(dx)\hat{\Psi}_{x0}^{-+},$$

where $\hat{\Psi}_{00}^{-+} = \lim_{b\to\infty} {}_0\Psi_{bb}^{-+}$, $\hat{\Psi}_{x0}^{++} = \lim_{b\to\infty} {}_0\Psi_{(b-x)b}^{+}$ and $\hat{\Psi}_{x0}^{-+} = \lim_{b\to\infty} {}_0\Psi_{(b-x)b}^{-+}$. From Proposition 2.1 and the definition of ${}_0\Psi_{xb}^{\bullet+}$, we have

$$\hat{\Psi}_{00}^{-+} = R_0, \quad \hat{\Psi}_{x0}^{++} = \exp(xU), \quad \hat{\Psi}_{x0}^{-+} = R_0\exp(xU).$$

Thus we have (I).

(Proof of (II)) By conditioning on the event that the MAP $(X(t), M(t))$ crosses level 0 from below, P_{00}^{++} is given by $H_0^{+-}R_0 + \int_0^\infty H^{+\bullet}(du)R^{\bullet+}(u)$, that is,

$$H_0^{+-}R_0 + \Delta_{\mathbf{r}^+}^{-1}\Delta_{\pi^+}^{-1}\left\{\int_0^\infty du\int_u^\infty \begin{pmatrix} \tilde{R}_0\exp((y-u)\tilde{U}) \\ \exp((y-u)\tilde{U}) \end{pmatrix}^{\mathrm{T}}\Delta_\pi D(dy)\begin{pmatrix} R_0 \\ I \end{pmatrix}\exp(uU)\right\}.$$

by Proposition 2.2. By postmultiplying $\mathbf{q}^+$, changing the order of integrations and $U\mathbf{q}^+ = \alpha\mathbf{q}^+$, we have

$$P_{00}^{++}\mathbf{q}^+ = \Bigg\{H_0^{+-}R_0 + \Delta_{\mathbf{r}^+}^{-1}\Delta_{\pi^+}^{-1}\int_0^\infty \begin{pmatrix} \tilde{R}_0\exp(y\tilde{U})\Delta_{\tilde{\mathbf{q}}^+}\int_0^y du\exp(-u\hat{U})\Delta_{\tilde{\mathbf{q}}^+}^{-1} \\ \exp(y\tilde{U})\Delta_{\tilde{\mathbf{q}}^+}\int_0^y du\exp(-u\hat{U})\Delta_{\tilde{\mathbf{q}}^+}^{-1} \end{pmatrix}^{\mathrm{T}}\Delta_\pi D(dy)\begin{pmatrix} R_0 \\ I \end{pmatrix}\Bigg\}\mathbf{q}^+, \tag{14}$$

where $\hat{U} = \Delta_{\tilde{\mathbf{q}}^+}^{-1}(\alpha I + \tilde{U})\Delta_{\tilde{\mathbf{q}}^+}$. Since $\tilde{U}$ also has the P-F eigenvalue $-\alpha$, denote the corresponding positive right eigenvector by $\tilde{\mathbf{q}}^+$, i.e., $\tilde{U}\tilde{\mathbf{q}}^+ = -\alpha\tilde{\mathbf{q}}^+$. Since $\hat{U}$ is non-defective transition rate matrix, denote its stationary distribution by κ. Then we have

$$\int_0^y du \exp(-u\hat{U}) = (\mathbf{1}\kappa - \hat{U})^{-1}(\exp(-y\hat{U}) + y\mathbf{1}\kappa - I). \tag{15}$$

From (14) and (15), we have (II).

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